POLYNOMIAL FUNCTIONS ON SUBRINGS OF MATRIX ALGEBRAS

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ABSTRACT. In this research, we study the null ideal of a polynomial ring $T(D)[x]$, where D is an integral domain and $T(D)$ is the ring of 3 by 3 tri-diagonal matrices with entries from D. The null ideal consists of the polynomials $f(x)$ in $T(D)[x]$ such that $f(A) = 0$ for all $A \in T(D)$. We give the structure of the null ideal and properties of the polynomials involved.

1. INTRODUCTION

Polynomials are not only central to a breadth of mathematical areas of study but also are fundamental to common applications of mathematics. Regarding real world applications, many scientific problems can be modeled by polynomial functions. In statistics, linear regression, quadratic regression, and higher degree polynomial regressions are often applied to describe complicated data structures through abstract and quantitative representations in the polynomial form. In many cases, solving a polynomial equation or finding all the roots of a polynomial is an important task in order to solve a challenging real-world problem. For example, the famous quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which serves as a powerful tool to find the solutions to the quadratic equation $ax^2 + bx + c = 0$, where a, b, c are arbitrary real numbers. Any solution to this equation gives the "zero" situation of the polynomial involved, which can be viewed as a root of the polynomial function $f(x) = ax^2 + bx + c$. Polynomials with real coefficients are considered one of the best types of functions among all the real functions because each of them is continuous and differentiable in any order everywhere along the real number line. Given any polynomial $f(x)$,

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the local maximum and minimum values of it only can be achieved at the roots of the derivative $f'(x)$. The concavity of the graph of $f(x)$ only changes at the root(s) of the second derivative $f''(x)$ and $f''(x)$ also tell the increasing or decreasing situation of the rate of change of the function $f(x)$ itself. It makes simple and feasible to solve any related optimization problem modeled by a polynomial.

Let us look at the problem from a different perspective. Consider two algebraic structures A and B, both having additive identity 0, and a map ϕ from A to B. The set $\ker(\phi) = \{a \in A \mid \phi(a) = 0\}$ is called the kernel of the map ϕ (see Definition 1.5). Consider any polynomial $f(x)$ of degree $n > 0$ with real coefficients. We can view it as a map, f, from the set $\mathbb{R}$ of all real numbers to itself. The the kernel of f is the set of all of the roots of $f(x)$, that is, $\ker(f) = \{a \in \mathbb{R} \mid f(a) = 0\}$. In particular, if $f(x) = ax^2 + bx + c$, a quadratic polynomial with real coefficients a, b, c, and let r_1 and r_2 be the two roots of $f(x)$ (it is possible that the two roots are identical, called double root). Then $f(r_1) = f(r_2) = 0$. If we define two related subsets of $\mathbb{R}$: $S_1 = \{r_1, r_2\}$ and $S_2 = \{0\}$, then the polynomial $f(x)$ maps every element in S_1 onto S_2. That is, $f(S_1) = S_2$ or $\ker(f) = S_1$. That the entirety of S_1 maps onto $\{0\}$, or S_1 as the kernel of the map f, is of importance here and forms the basis of inquiry into this particular research.

There are two binary operations in the real number system, the addition "+" and the multiplication "$\times$". These two operations satisfy a set of axioms such as commutativity, closure of both operations, and distributive properties. One of such algebra structures is called a ring, which is of the interest for this research. We focus on the ring $\mathbb{R}$ of all real numbers, the ring $\mathbb{Z}$ of all integers, the ring $\mathbb{R}[x]$ of all real polynomials, and the ring of certain matrices. The precise definition for a set to be a ring is given in section 1.1.

Now consider a commutative ring R. An important type of subset of R is called "ideal" of R, which is a subring of R satisfying additional conditions (see Definition 1.3). Briefly, if I is an ideal of R, then it is closed under the addition and multiplication of R (sum and product sit in I) and furthermore, when multiplying an element of R, which may be outside of I, with an element in I, the product is in I. In case R is a noncommutative ring, "left ideals" and "right ideals" correspondingly defined. The popular "ideal membership problem" is to determine whether a given subset of R is an ideal of R and it attracts many researchers in ring theory. Consider two non-empty subsets S_1, S_2 of a ring R and denote the set of all polynomials in $R[x]$ that map S_1 into S_2 by

$$\text{Int}(R)_{(S1,S2)} = \{f(x) \in R[x] \mid f(S_1) \subset S_2\}.$$

A natural question is that, is the set $\text{Int}(R)_{(S_1,S_2)}$ an ideal of $R[x]$? In general, the answer is "no". Though, it is the case for certain polynomial rings that the answer is "yes" [3], meaning it claims the ideal membership. An example is given below.

Example 1.1. *Let $f(x) = x^2 - 3x + 6$ and $g(x) = 3x^2 - 9x + 10$ be two polynomials in $\mathbb{R}[x]$. It is easy to see that both $f(x) - 4$ and $g(x) - 10$ have two roots: 1 and 2. Consider two subsets of $\mathbb{R}$, $S_1 = \{1, 2\}$ and $S_2 = \{4\}$. Then $f(1) = 4, g(x) = 4$ and so $f(x), g(x) \in Int(\mathbb{R})_{(S_1,S_2)}$. But $f(x) + g(x) \notin Int(R)_{(S_1,S_2)}$ because $f(1) + g(1) = 8 \notin S_2$. Thus $Int(\mathbb{R})_{(S_1,S_2)}$ in not an ideal of $\mathbb{R}[x]$.*

However, if we take $S_1 = \{1, 2\}$, $S_2 = \{0\}$, then $Int(\mathbb{R})_{(S_1,S_2)}$ is an ideal of $\mathbb{R}[x]$.

Consider a ring R. In this paper we focus on a related problem that is important to mapping such sets. The problem at a more general level is to identify some subsets S_1 into S_2 of the ring R such that the set of all polynomials in $R[x]$ mapping S_1 into S_2 is an ideal of $R[x]$. Then the main question becomes: "Is the set $\text{Int}(R)_{(S_1,S_2)}$ an ideal of $R[x]$? The background knowledge on this topic is the integer-valued polynomial problem which deals with a special case of the above, when $S_1 = S_2$. Suppose D is an integral domain with quotient field K (Definition 1.4). With the setting of $S_1 = D = S_2 \subseteq D$, we obtain a new set $\text{Int}(D) = \{f(x) \in K[x] \mid f(D) \subset D\}$. This set is called the set of integer-valued polynomials over D. It is known that $\text{Int}(D)$ is a subring of the polynomial ring $K[x]$.

The ring $\text{Int}(D)$ has been extensively studied, for instance in [2]. The name "integer-valued polynomial" suggests that it is a generalization of the "polynomial with integer coefficients". Indeed, if $D = \mathbb{Z}$, then $K = \mathbb{Q}$, the ring of rational numbers. One can check that the polynomial $f(x) = x + 1$ maps all integers to integers. Thus, $f(x) \in \mathrm{Int}(\mathbb{Z})$. However, as stated in [2], the integer-valued polynomials over a general algebraic structure are largely still undeveloped. More recently attention has been given to the case over a matrix ring. In [11], the notion of a two sided ideal is considered with regard to the null ideal as defined later in Definition 1.17. Some general results for matrices in $M_n(D)$ (the ring of $n \times n$ matrices over D), where D is a domain, are given along with some specific examples of up to subrings of $M_3(D)$. Here we start by considering a special subset of matrices in $M_3(D)$ which is known to form a ring.

We are interested in a special type of ideals, called "null" ideal defined in Definition 1.17. In particular, we focus on the null ideal of a polynomial ring $T[x]$, where T is a subring of the matrix ring $M_n(D)$, where D is an integral domain. We discuss what conditions are needed to be added so that a given set of matrices can form a subsring. A set of 3×3 tri-diagonal matrices, denoted $T(D)$, where D is a domain, is identified to be our focus. Determining whether a null ideal of a subring of $M_n(R)$ is two-sided takes considerable work as shown in [11]. Here we focus on the left null ideals as left ideals of $T(D)$. The methodology used is more direct analysis of the product of two members of $T(D)$. In contrast, the ground ring under study in this research is an integral domain, whereas in [11],[12], the ground ring is a field. The difference is significant as some assumptions cannot be made from the outset when determining the null ideal of a particular subset. It may be of interest to discuss subsets of these matrices which are not necessarily subrings of $M_n(D)$, but here we will focus on subrings of $M_n(D)$. First we will show a few basic definitions .

1.1. **Ring Theory Basics.**

Definition 1.2 ([1, p. 96])**.** *A set R with two binary operations (typically denoted $\cdot$ and $+$) and a corresponding identity for each operation (1 and 0 here respectively) is called a ring if it satisfies the following properties. Let $x, y, z \in R$.*

(1) $x + (y + z) = (x + y) + z$

(2) $x + y = y + x$
(3) $0 + x = x + 0 = x$
(4) $x + (-x) = (-x) + x = 0$
(5) $x(yz) = (xy)z$
(6) $x \cdot 1 = 1 \cdot x = x$
(7) $(x + y)z = xz + yz$
(8) $x(y + z) = xy + xz$

If $xy = yx$ we say that R is commutative ring.

One of the most important defining properties of a ring is that certain subsets of a ring exist called *ideals* defined in the following way.

Definition 1.3. *An ideal I is a subset of a ring R if for any element $x \in I$ and $y \in R, xy \in I$. In a noncommutative ring, if $xy \in I$ and $yx \in I$, then I is called a two-sided ideal.*

Given some element $x \in R, R$ a ring, we call the ideal $(x) = xR$ a *principal ideal* that is generated by A *zero divisor* of a ring R is an element $x \in R, x \neq 0$, if there exists a $y \in R$ such that $xy = 0$. We use this occurrence in some rings to define a particular type of ring for which this property does not hold.

Definition 1.4. *A ring D is called an integral domain if it has no zero divisors. A field is an integral domain where every element has a multiplicative inverse, i.e. an element $x \in F$ where F is a field and $x \neq 0$, there is some $y \in F, y \neq 0$ such that $xy = 1$.*

$\mathbb{Z}$ is an integral domain since 0 is the only zero divisor, but not a field since we have that the only invertible element is 1. In order to show that $\mathbb{Q}$ is a field over $\mathbb{Z}$, we introduce the notion of a map between two rings which is well defined and structure preserving called a *ring homomorphism*. When we use the term homomorphism in the context of this paper it is implied that it is a ring homomorphism unless otherwise explicitly stated. These maps define a relationship between two different rings, though they may define a relationship from a ring to itself.

Definition 1.5. *Let R_1 and R_2 be rings. If $x, y \in R_1$, then a map T which is denoted by $T : R_1 \to R_2$ is called a homomorphism if it has the following properties:*

(1) $T(xy) = T(x)T(y)$.
(2) $T(x + y) = T(x) + T(y)$.

The *preimage* of T is denoted as $T^-1(x), x$ an element of R_2. The *kernel* of a homomorphism Ker(T)=$\{y \in T^{-1}(0) \mid y \in R_1\}$. In other words, the kernel of a homomorphism is all elements $y \in R_1$ for which $T(y) = 0$.

If for all $x \in R_1$ there is some unique element $y \in R_2$ such that $T(x) = y$, the homomorphism T is considered to have an *injective* relationship. If for every unique $y \in R_2$ there is some unique $x \in R_1$ such that $T(x) = y$ then T is considered *surjective*.

Definition 1.6. *If T is both injective and surjective then T is called bijective. A bijective homomorphism is called an isomorphism.*

1.2. **Localization of a Ring.**

In addition to the basic ring definitions, we want to define a special structure which can be constructed over every integral domain called its *field of fractions*. We require this structures definition to further define the rings of polynomials we will use to discuss the problem at hand. A *multiplicative set S* is a subset of a ring which includes the unit element and is multiplicatively closed, i.e., for any elements r, s in the set, $rs \in S$. We use the general form from Definition 1.7 (3) and disregard a multiplicative set which includes 0. This is the case because when $0 \in S$ then for any $\frac{r}{s}, r \in R, s \in S, \frac{r}{s} = \frac{r}{0}$ since $0(r0 - rs) = 0$, hence, the entire ring is equal to 0. Then we define S to be a multiplicative set excluding 0.

If R is a ring and $S = R \setminus \{0\}$, then let $S^{-1}R = \{\frac{r}{s} \mid r \in R, s \in S\}$ which is the *total ring of fractions* of R. If R is an integral domain then this is an equivalent definition for the field of fractions over R. Let $\phi : R \to S^{-1}R$ be the canonical map defined $r \mapsto \frac{r}{1}$. This homomorphism is quite obviously injective due to there being no zero divisors in S. We may generate $S^{-1}R$ from S^{-1} and ϕ if we take the product of elements of S^{-1} and every element in $\phi(R)$. Now every element of $S^{-1}R$ has an inverse and since there are no zero divisors, $S^{-1}R$ is a field. We have in a sense filled out R and extended it to a ring which "completes" it in the sense of transforming it into a field. More generally, $S^{-1}R$ is called the localization of R at S if S is any multiplicative subset of R. When we consider R to be any ring, then it is possible our choice of S will render $S^{-1}R$ to not be a field. More formally:

Definition 1.7. *Let D be an integral domain and $q, r, s, t, \in D$, $\frac{q}{r}, \frac{s}{t} \in K$. The following are true of K.*

(1) $\frac{q}{r} + \frac{s}{t} = \frac{qt+rs}{rt}$
(2) $\frac{q}{r} \cdot \frac{s}{t} = \frac{qs}{rt}$
(3) $\frac{q}{r} = \frac{s}{t} \iff qt = rs$.

More generally for (3) we say that $\frac{q}{r} = \frac{s}{t} \iff \exists v \in D$ such that $v(qt-rs) = 0$.

Example 1.8. *$\mathbb{Q}$ is the field of fractions over $\mathbb{Z}$ by letting the elements in the numerator be $\mathbb{Z}$ and the elements in $\mathbb{Z} \setminus \{0\}$ be in the denominator.*

Throughout, for two non-negative integers m and n with $m < n$, $[\![n, m]\!] = \{m, m+1, \ldots, n\}$. We denote the nilradical of a ring R to be $\mathfrak{N} = \{x \in R \mid x^n = 0\}$.

1.3. **Polynomial Rings and Integer-valued Polynomial Rings.**

For our treatment of the algebraic structures we discuss in this paper we require the definition of rings formed over indeterminates.

Definition 1.9. *Let R be a ring. Then we call the ring denoted $R[x]$ the polynomial ring over a ring R with indeterminate x.*

The above definition may also be expanded to define polynomials of a multivariate type which is denoted $R[x_1, \ldots, x_n]$ up to n indeterminates. The coefficients of a polynomial ring are typically from the same ground ring that the indeterminates cover, but in some cases we may define a set of polynomials with indeterminate $x \in R$ but with coefficients in A where, $R \subseteq A$. In fact the motivation for the study of null ideals in this paper originates from the following definition.

Definition 1.10. *Let R be an integral domain and K its field of fractions. Then,*

$$Int(R) = \{f(x) \in K[x] \mid f(R) \subseteq R\}$$

is the set of integer-valued polynomials over R.

There is a variety of notation we will use to mean integer-valued polynomials given subsets of the polynomial ring in question.

Definition 1.11 ([5])**.** *Let D be an integral domain with quotient field K and A is an ideal of D or $T(D)$. Then*

$$\begin{aligned} Int(D) &= \{f(x) \in K[x] \mid f(D) \subseteq D\}; \\ Int_D(T(D)) &= \{f(x) \in K[x] \mid f(T(D)) \subseteq T(D)\}; \\ Int_D(A) &= \{f(x) \in K[x] \mid f(A) \subseteq A\}; \\ Int_D[T(D)] &= \{f(x) \in T(D)[x] \mid f(T(D)) \subseteq T(D)\}. \end{aligned}$$

The polynomials we will be discussing in this research are polynomials of which their indeterminates, as well as their coefficients, are members of a matrix ring defined over an integral domain.

1.4. **Matrix Rings.**

A *matrix* is an arrangement of elements from a ring that are arranged in an $m \times n$ grid, $m, n \in \mathbb{N}$. We will restrict our discussion to matrices of $n \times n$ size in this paper. The ring of $n \times n$ size square matrices is denoted $M_n(R)$ where the elements that populate each matrix are from R. We give the following operations for matrices. We use lowercase letter corresponding to the named matrix with two natural numbers as their index to indicate the ij-th element of some matrix, i.e., given some matrix $A \in M_n(R)$, the ij-th element is denoted as a_{ij}. We also use the notation $[A]_{ij}$ interchangeably. $M_n(R)$ is a ring under special operations of addition and multiplication.

Definition 1.12. *Let $A, B \in M_n(R)$ and R a ring. Then for every element $a_{ij} \in A$ and $b_{ij} \in B$,*

(1) *$A + B$ is defined such that $[A + B]_{ij} = a_{ij} + b_{ij}$.*
(2) *AB is defined such that $[AB]_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.*

The zero element is the matrix contained in the set of matrices where every element is zero, which we will denote by a bold $\mathbf{0}$ without further explanation when context is clear. The identity element in $M_n(R)$ is denoted I where every diagonal element is 1 and every other element is 0. For any matrix $A \in M_n(R), IA = AI = A$.

Example 1.13. *The identity matrix for $M_3(R), R$ a ring, is*

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example 1.14. *$M_3(K)$ is the ring of 3×3 matrices over a field K.*

The identity element, zero element, and operations of multiplication and addition have the same properties as they do in any noncommutative ring. A subset $S \subset M_n(R)$ may also be a ring if it includes I. An important aspect of the problem given here is that we must choose a subset of matrices in $M_n(R)$ which are also rings so that they may fit the requirements by the definition of an integer-valued polynomial. We will now define a subset of $M_n(R)$ which has been studied previously by the researchers, and fits the requirements for a matrix subring quite nicely.

1.5. **The Matrix Ring $T(R)$.**

Definition 1.15. *Let R be a ring. We define a subset $T(R) \in M_n(R)$ to have the following form. For any $A \in T(R)$,*

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

Given this definition we now show that $T(R)$ is a ring when R is commutative.

Lemma 1.16. *Let R be a commutative ring. $T(R)$ is a ring.*

Proof. Let $A, B \in T(R)$. Addition is obviously closed. Multiplication is as well:

$$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix}$$
$$= \begin{pmatrix} a_{11}b_{11} & 0 & 0 \\ a_{21}b_{11} + b_{21}a_{22} & a_{22}b_{22} & a_{22}b_{23} + a_{23}b_{33} \\ 0 & 0 & a_{33}b_{33} \end{pmatrix}$$

□

A simple fact is that the determinant of a matrix A in $T(R)$, $|A| = a_{11}a_{22}a_{33}$. The set $T(R)$ forms a (noncommutative) matrix ring.

Let R be a commutative ring and set $S = R$ and $I = (0)$.. Consider the subring $U_n(R)[x]$ of the $n \times n$ upper-triangular matrices of $M_n(R)[x]$. Two kinds of "null-polynomial functions" on upper triangular matrices are investigated in [3], those induced by polynomials with matrix coefficients on the one hand, and those induced by polynomials with scalar coefficients on the other. They are defined below:

$$\mathrm{Int}_R(U_n(R), 0) = N_R(U_n(R)),$$
$$\mathrm{Int}_{U_n(R)}(T_n(R), 0) = N_{U_n(R)}(U_n(R)).$$

Here $\mathbf{0} = \{0_{n\times n}\}$ is the set of one member, the $n \times n$ zero matrix. In general, for any given ring R' and a subring S' of R', the set $N'_S(R') = \{f(x) \in S'[x] \mid f(a) = 0 \text{ for all } a \in R'\}$ is the set of all polynomials with coefficients in S' that maps every element in R' to $\mathbf{0}$, or it is called the set of "null-polynomials" on R'. One of the results of [3] is the claim that the subrings $N_R(U_n(R))$ and $N_{U_n(R)}(U_n(R))$ are ideals of the polynomial rings $R[x]$ and $U_n(R)[x]$ respectively.

The aim of this paper is to generalize these results of the work in [3] to other subrings of $M_n(R)$. In particular, we aim to identify subring T of $M_n(R)$ such that $N_{T[x]}(T[x])$ is an ideal of $T[x]$. One consideration is to start with the following setting: select some fixed subset S of transformations $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ and let T be the set of matrices in the form of (a_{ij}) with $a_{ij} = 0$ for all (i, j) in S and the other entries being arbitrary. This selected set S

should make the resulting set T to be a subring of $M_n(R)$. We will examine different settings for S and the resulting T, and study properties of T and the null polynomials of T.

The ring $T(R)$ defined in Definition 1.15 is such a ring that meets this criteria. It is easy to see if we let $S = \{(1,2),(1,3),(3,1),(3,2)\}$, then for every $(i,j) \in S$ and $A \in T(R), [A]_{ij} = 0$.

Definition 1.17. *Let R be an integral domain with quotient field K. Assume $S \subseteq R$. The null ideal of S in R is given by*

$$N_R(S) = \{f(x) \in R[x] \mid \ \forall s \in S, f(s) = 0\}.$$

Special attention is given to $N_R(R)$, the case when $S = R$, called the null ideal of R. Regarding to the matrix ring $T(R)$ defined above, two related ideals are also to be discussed in this paper.

2. Current Research and Goals

The topic of null ideals is well related to that of integer-valued polynomials by a residue class ring of a domain [12]. There is a body of growing literature with an aim to examine integer-valued polynomials over rings which are noncommutative, and more specifically, matrices of varying sizes and type [2][3][6][8][9][10][7][11][12]. Strong results for upper triangular matrices exist [3], from which this paper is motivated initially. An important question for which current research intends to develop an answer is: given some integral domain D, for which subsets $S \subseteq M_n(D)$ does $\mathrm{Int}(S, M_n(D))$ a ring?. It is known that if $I \subset D$ is an ideal, then $\mathrm{Int}(M_n(I), M_n(D))$ is a ring [4]. However, for the general case of any subset of a matrix ring, the conditions for which its integer-valued polynomials form a ring is not clear. One way of approaching the problem is to determine the null ideal of a subset of a ring. The reason why can be shown by formulating the integer valued polynomials as a residue class of polynomials in $D[x]$ over domain D.

This relationship is shown in [11]. Given some $f \in M_n(D)$, we may rewrite $f = g(x)/d$, which belongs to some residue class ring in $M_n(D)[x]$. Let (d) be the principal ideal generated by dD, and $g' \in M_n(D/(d))[x]$. Then for all $A \in S \subset M_n(D), f(A) \in M_n(D)$ if and only if $g(A) \in M_n(dD)$ if and only if $g'(A') = 0$ in $M_n(D/(d))$. The set of polynomials for which $g'(A') = 0$ is exactly the null ideal of S, thus determining the set of polynomials in the null ideal gives a complete description of $\mathrm{Int}(S, M_n(D))$. We now have motivation for the study of these particular types of rings and their null ideals.

We have already introduced the special type of ring $T(D)$ above. Here we look at $N_{T(D)}(T(D))$ and this notation is shortened to $N(T(D))$ when $S = D$, in the definition. Then $N(T(D) = \{f \in T(D)[x] \mid f(X) = 0 \text{ for all } X \in T(D)\}$. It has already been determined that the subset $T(D) \subset M_n(D)$ is a ring. The aim is to now determine what criteria must be met by some polynomial in $T(D)[x]$ to be included in the null ideal $N_{T(D)}(T(D))$. As it turns out, determining this relationship is not simple or obvious. In section 3 we provide verification that such an ideal exists, section 4 contains some examples, section 5 discusses what other types of rings could be studied, and section 6 concludes with some possible directions for future research.

3. Basic Operations

We prefer first to start by determining the powers of a matrix in $T(D)$ by functions which generate the elements at each position in the resulting matrix.

Lemma 3.1. *Let R be an integral domain and $T(D)$ as defined above. Consider $X = [x_{ij}] \in T(D)$. Then for any integer $n \geq 1$,*

$$X^n = \begin{pmatrix} x_{11}^n & 0 & 0 \\ x_{21}\sum_{i=0}^{n-1} x_{11}^i x_{22}^{n-1-i} & x_{22}^n & x_{23}\sum_{i=0}^{n-1} x_{22}^i x_{33}^{n-1-i} \\ 0 & 0 & x_{33}^n \end{pmatrix}.$$

Proof. Obviously, the formula is true for $n = 1$. For $n > 1$, it can be shown by mathematical induction on n with the following calculation:

$$X \cdot X^{n-1} =$$
$$\begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix} \begin{pmatrix} x_{11}^{n-1} & 0 & 0 \\ x_{21}\sum_{i=0}^{n-2} x_{11}^i x_{22}^{n-2-i} & x_{22}^{n-1} & x_{23}\sum_{i=0}^{n-2} x_{22}^i x_{33}^{n-2-i} \\ 0 & 0 & x_{33}^{n-1} \end{pmatrix}$$
$$\begin{pmatrix} x_{11}^{n} & 0 & 0 \\ x_{21}\sum_{i=0}^{n-1} x_{11}^i x_{22}^{n-1-i} & x_{22}^{n} & x_{23}\sum_{i=0}^{n-1} x22^i x_{33}^{n-1-i} \\ 0 & 0 & x_{33}^{n} \end{pmatrix}.$$

□

Given a polynomial in $T(D)[x]$, by Lemma 3.1 and matrix operations, we can evaluate it at any matrix in $T(D)$. The formula is shown below:

Lemma 3.2. *Let $f(x) = A_n x^n + \cdots + A_1 x + A_0 \in T(D)[x]$, the polynomial ring in one variable over the ring $T(D)$, where $A_k = [a_{ij}^{(k)}] \in T(D)$. For any matrix $X = [x_{ij}] \in T(D)$,*

$$f(X) = \begin{pmatrix} \sum_{k=0}^n a_{11}^{(k)} x_{11}^k & 0 & 0 \\ g_1(x_{11}, x_{21}, x_{22}) & \sum_{k=0}^n a_{22}^{(k)} x_{22}^k & g_2(x_{22}, x_{23}, x_{33}) \\ 0 & 0 & \sum_{k=0}^n a_{33}^{(k)} x_{33}^k \end{pmatrix},$$

where

$$\begin{aligned} g_1(x_{11}, x_{21}, x_{22}) &= \sum_{k=1}^{n} \left[a_{21}^{(k)} x_{11}^k + x_{21} a_{22}^{(k)} \sum_{i=0}^{k-1} x_{11}^i x_{22}^{k-1-i} \right], \quad \textit{and} \\ g_2(x_{22}, x_{23}, x_{33}) &= \sum_{k=1}^{n} \left[a_{23}^{(k)} x_{33}^k + x_{23} a_{22}^{(k)} \sum_{i=0}^{k-1} x_{22}^i x_{33}^{k-1-i} \right]. \end{aligned}$$

Equipped with

Theorem 3.3. *Let $f(x) = A_n x^n + \cdots + A_1 x + A_0 \in T(D)[x]$, the polynomial ring in one variable over the ring $T(D)$, where $A_k = [a_{ij}^{(k)}] \in T(D)$. Then $f(X) \in N_{T(D)}(T(D))$ if and only if*

$$\forall (i,j) = (1,1), (2,1), (2,2), (2,3), (3,3), \ \sum_{k=1}^{n} a_{ij}^{(k)} x^k \in N_D(D).$$

Proof. Denote $\vec{x} = (x_{11}, x_{21}, x_{22}, x_{23}, x_{33})$, the vector of the five involved variables over D. By Lemma 3.2, We further write

$$\begin{aligned} f(X) &= \begin{pmatrix} f_{11}(\vec{x}) & 0 & 0 \\ f_{21}(\vec{x}) & f_{22}(\vec{x}) & f_{23}(\vec{x}) \\ 0 & 0 & f_{33}(\vec{x}) \end{pmatrix} \\ &= \begin{pmatrix} f_{11}(x_{11}) & 0 & 0 \\ g_1(x_{11}, x_{21}, x_{22}) & f_{22}(x_{22}) & g_2(x_{22}, x_{23}, x_{33}) \\ 0 & 0 & f_{33}(x_{33}) \end{pmatrix}. \end{aligned}$$

Then for $i = 1, 2, 3$, $f_{ii}(x_{ii}) = \sum_{k=1}^{n} a_{ii}^{(k)} x_{ii}^{k} \in D[x_{ii}]$ (constant must be 0). Note that x_{ii} is treated as an indeterminate over R.

Now assume $f \in N_{T(D)}(T(D))$. Then $f(X) = 0$ for all $X \in T(D)$. It implies for each $i = 1, 2, 3$, $f_{ii}(a) = 0$ for all $x_{ii} := a \in D$, that is, $f(D) = 0$. Thus, $f_{ii} \in N_D(D)$. By lemma 3.2,

$$f_{21}(\vec{x}) = g_1(x_{11}, x_{21}, x_{22}) = \sum_{k=1}^{n} \left[a_{21}^{(k)} x_{11}^{k} + x_{21} a_{22}^{(k)} \sum_{i=0}^{k-1} x_{11}^{i} x_{22}^{k-1-i} \right].$$

Because $f(X) = 0$ for all $X \in T(D)$, $g_1(a, b, c) = 0$ for all $a, b, c \in D$. By substituting $x_{21} = 0$, we have $h_1(x_{11}) = \sum_{k=1}^{n} a_{21}^{(k)} x_{11}^{k} = 0$ for all values $x_{11} \in D$. Thus, $h_1 \in N_R(D)$. It further requires that $g_1(x_{11}, x_{21}, x_{22}) - h_1(x_{11}) = 0$ for all values of x_{11}, x_{21}, and x_{22} in D. That is,

$$\forall x_{11}, x_{21}, x_{22} \in T(D), \quad x_{21} \sum_{k=1}^{n} a_{22}^{(k)} \left[\sum_{i=0}^{k-1} x_{11}^{i} x_{22}^{k-1-i} \right] = 0.$$

It is sufficient to examine the above for all nonzero values of x_{11}, x_{21}, x_{22}. Then we need

$$\sum_{k=1}^{n} a_{22}^{(k)} \left[\sum_{i=0}^{k-1} x_{11}^{i} x_{22}^{k-1-i} \right] = 0 \quad \text{for all values of } x_{21}, x_{22} \text{ in } D \setminus \{0\}.$$

In case $x_{11} = 1 = x_{22}$, we obtain

$$\sum_{k=1}^{n} a_{22}^{(k)} \left[\sum_{i=0}^{k-1} x_{11}^{i} x_{22}^{k-1-i} \right] = \sum_{k=1}^{n} a_{22}^{(k)} \cdot k = f'_{22}(1).$$

In case one of the x_{11} and x_{22} is 1 but the other is neither 0 nor 1, say, $x_{11} \neq 0, 1$ but $x_{22} = 1$, we have

$$\sum_{k=1}^{n} a_{22}^{(k)} \left[\sum_{i=0}^{k-1} x_{11}^{i} \right] = \frac{1}{x_{11}-1} \sum_{k=1}^{n} a_{22}^{(k)} \left(x_{11}^{k} - 1 \right)$$
$$= \frac{1}{x_{11}-1} \sum_{k=1}^{n} a_{22}^{(k)} x_{11}^{k} - \frac{1}{x_{11}-1} \sum_{k=1}^{n} a_{22}^{(k)}.$$

Since $f_{22}(x) = \sum_{k=1}^{n} a_{22}^{(k)} x^k \in N_D(K) \subseteq N_D(D)$, we have both $\sum_{k=1}^{n} a_{22}^{(k)} x_{11}^{k} = 0$ and $\sum_{k=1}^{n} a_{22}^{(k)} = f_{22}(1) = 0$.

Finally, we consider the case when $x_{21}, x_{22} \neq 0, 1$. In the field K of fractions of D, with

$$\sum_{k=1}^{n} a_{22}^{(k)} \left[\sum_{i=0}^{k-1} x_{11}^{i} x_{22}^{k-1-i} \right] = \sum_{k=1}^{n} a_{22}^{(k)} \left[\frac{1}{x_{11}-x_{22}} \left(x_{11}^{k} - x_{22}^{k} \right) \right]$$
$$= \frac{1}{x_{11}-x_{22}} \sum_{k=1}^{n} a_{22}^{(k)} x_{11}^{k} - \frac{1}{x_{11}-x_{22}} \sum_{k=1}^{n} a_{22}^{(k)} x_{22}^{k} = 0.$$

With the same reason, $f_{22}(x) = \sum_{k=1}^{n} a_{22}^{(k)} x^k \in N_D(D)$, thus, $\sum_{k=1}^{n} a_{22}^{(k)} x_{11}^{k} = 0$ and $\sum_{k=1}^{n} a_{22}^{(k)} x_{22}^{k} = 0$. Similar properties hold for the (2,3)-entry of $f(X)$, $g_2(x_{22}, x_{23}, x_{33})$.

In summary, $f(X) \in N_{T(D)}(T(D))$ if and only if all the five polynomials as the entries of $f(X)$ must be in the null ideals $N_D(D)$. That is, $\forall x_{11}, x_{21}, x_{22} \in T(D)$, $\quad x_{21} \sum_{k=1}^{n} a_{22}^{(k)} \left[\sum_{i=0}^{k-1} x_{11}^{i} x_{22}^{k-1-i} \right] = 0.$ $\square$

Corollary 3.4. *Let D be any integral domain. The null ideal $N_{T(D)}(T(D)$ is given by*

$$N_{T(D)}(T(D)) = \begin{pmatrix} N_D(D) & 0 & 0 \\ N_D(D) & N_D(D) & N_D(D) \\ 0 & 0 & N_D(D) \end{pmatrix}.$$

Trivially, the null ideal of a ring is two-sided. Null ideals of subsets of a ring, however, are not guaranteed, and finding them is the topic explored in [11]. We will briefly observe this in the special matrices $T(D)$ later in the paper. Since we restrict our set of polynomials by the entire ring, we get the following corollary for free.

Corollary 3.5. *The null ideal $N_{T(D)}(T(D))$ is two-sided.*

4. Properties of Null Polynomials

In this section, we give specific properties of polynomials in the null ideal $N_{T(D)}(T(D))$. We may also examine situations when R is a finite field like $R = \mathbb{Z}_p$, where p is a prime. For small prime numbers, we can give a complete picture of the null ideal (list all the elements). Let the reader remind themselves that the units of a polynomial ring $D[x]$ where D is an integral domain, is $D \subset D[x]$. This informs the exclusion of polynomials which have a constant term in the following propositions.

Lemma 4.1. *Let $R = \mathbb{Z}_3$ and $f(x) = a_n x^n + \cdots + a_1 x \in \mathbb{Z}_3[x]$, $n > 0$. Then*

(1) *if n is even, $f(x) \in N_R(R)$ if and only if*

$$\sum_{i=1}^{n/2} a_{2i} = 0 = \sum_{i=1}^{n/2} a_{2i-1}.$$

(2) *If n is odd, then $f(x) \in N_R(R)$ if and only if*

$$\sum_{i=1}^{(n-1)/2} a_{2i} = 0 = \sum_{i=1}^{(n+1)/2} a_{2i-1}.$$

Lemma 4.1 also holds for the case of $\mathbb{Z}_2$, though the case for $\mathbb{Z}_2$ is trivial.

Corollary 4.2. *Let $R = \mathbb{Z}_2$. Then, $f(x) \in N_R(R)$ if and only if $f(x) = a_n x^n + \cdots a_1 x \in \mathbb{Z}_2, n > 0$, and has an even number of terms.*

Proof. The only case to consider is $x = 1$. If the number of terms with nonzero coefficient is even, then $2 \mid f(1)$ and $f(1) = 0$.

Conversely, assume $f(x) \in N_R(R)$. If $x = 0$ and $f(x) = a_n x^n + \ldots + a_1 x + 1$, then $f(1) \neq 0$, so $f(x)$ must be of the form $a_n x^n + \ldots + a_1 x$. Also, if $x = 1$ then $2 \nmid f(1)$ and $f \notin N_R(R)$. □

5. Characterization of Subrings of $M_n(R)$

Let R be a commutative ring and $\emptyset \neq S \subseteq M_n(R)$. The big question to answer in this section is that is S a subsring of $M_n(R)$? It is known that the two trivial subsets, $M_n(R)$ and $\{0\}$, are subrings. The set of all upper triangular matrices and the set of all diagonal matrices are both subrings. We look for other subrings. When $n = 3$, the set $T(R)$ introduced before is a subring of $M_3(R)$ (Lemma 1.16), where 6 of the entries in any matrix in $T(R)$ must be always 0. Another subset also forms a which posses a minimum of 4 zero in the entries. We format these two subsets in the following example.

Example 5.1. *Let R be a commutative ring. The two subsets, $T(R), W(R)$ of $M_3(R)$ are subrings:*

$$T(R) = \begin{pmatrix} R & 0 & 0 \\ R & R & R \\ 0 & 0 & R \end{pmatrix} \quad \text{and} \quad W(R) = \begin{pmatrix} R & 0 & R \\ R & R & R \\ R & 0 & R \end{pmatrix}.$$

Here an entry with R means any number from R are allowed in the position.

It indicates that the number of 0's and the positions of the 0's may contribute for the subset to be a subring. We next define a zero-index set and a zero-index for a subset of matrices.

Definition 5.2. *Let R be a commutative ring and $S_n(R) \subseteq M_n(R)$. The zero-index set of $S_n(R)$ is defined as*

$$Id_0(S_n(R)) = \{(i,j) \mid 0 \le i,j \le n, \text{ the } (i,j)\text{-entry of } A \text{ is } 0 \; \forall A \in S_n(R)\}.$$

The cardinality $|Id_0(S_n(R))|$ is called the zero-index of $S_n(R)$ and is denoted as $ind(S_n(R))$.

Immediately from the above definition, we have

Example 5.3. *Consider the two sets $T(R)$ and $W(R)$ discussed above.*

$$Id_0(T(R)) = \{(1,2),(1,3),(3,1),(3,2)\} \quad \text{and} \quad Id_0(W(R)) = \{(1,2),(3,2)\}.$$

The two zero-indices are 4 and 2 respectively.

Proposition 5.4. *Let R be a commutative ring and n be a positive integer. If we denote the set of all $n \times n$ upper triangular matrices over R by $U_n(R)$, then*

$$Id_0(U_n(R)) = \{(i,j) \mid 0 \le i,j \le n \text{ and } i > j\} \quad \text{and} \quad ind(U_n)R)) = \frac{n(n-1)}{2}.$$

The arrangement of zero's determines whether a set of matrices is a ring, along with the normal criteria for a ring. Next we define a subset of matrices and later prove that it forms a ring.

Definition 5.5.

$$L_n(R) := \{A \in M_n(R) \mid a_{ij} = 0, i \textit{ odd and } j \textit{ even}\}.$$

Note that when $n = 3$ the set in $L_3(R) = W(R)$. What remains to show is that this subset $L_n(R)$ of $M_n(R)$ is a ring itself.

Proposition 5.6. *$L_n(R)$ is a ring.*

Proof. All conditions for $L_n(R)$ to be a subring barring the multiplicative property follow from being a subset of $M_n(R)$. Let $A = [a_{ij}], B = [b_{ij}] \in L_n(R)$ and $AB = [c_{ij}]$. Then when i is odd and j is even,

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k\geq 2 \text{ and } k \text{ is even}} a_{ik}b_{kj} + \sum_{k\geq 1 \text{ and } k \text{ is odd}} a_{ik}b_{kj} = 0$$

because in the first summasion, $a_{ik} = 0$ and in the second summasion, $a_{kj} = 0$. Thus, $AB \in L_n$. □

The result in Proposition 5.6 produces an easily verifiable consequence with regard to subrings of $L_n(R)$.

Corollary 5.7. *$T(R)$ is a subring of $L_3(R)$.*

The zero ring has the maximal zero-index (n^2). The set of all diagonal matrices, a non-trivial subring of $M_n(R)$, has the zero-index $n^2 - n$. The zero-index of $L_n(R)$ can be calculated using combinatorial methods and we hope it can be used as a springboard to characterize all the subrings of $M_n(R)$.

Proposition 5.8. *The zero-index of $L_n(R)$ is $ind(L_n(R)) = \left\lfloor \frac{n^2}{4} \right\rfloor$. That is, $ind(L_n(R)) = n^2/4$ if n is even and $(n^2-1)/4$ when n is odd.*

Proof. When n is even, there are $n/2$ rows with the first index odd (rows $1, 3, \ldots, n-1$) and $n/2$ columns with the second index even (columns $2, 4, \ldots, n$). These positions have 0 entries and there are $(n/2)^2 = n^2/4$ many such positions. Similarly, when n is odd, there are $(n + 1)/2$ odd rows and $(n - 1)/2$

even columns, which result in $((n+1)/2)(n-1)/2 = (n^2-1)/4$ positions for 0 entries. $\square$

As an example, let p be a prime number and $\mathbb{F}_p$ be a finite field with p elements. For each matrix in $A \in L_n(F_p)$, it must have at least $\left\lfloor \frac{n^2}{4} \right\rfloor$ zero entries and each of the other entries can be chosen from the p elements of $\mathbb{F}_p$. Thus, the size of $L_n(\mathbb{F}_p)$ is

$$|L_n(\mathbb{F}_p)| = p^{n^2 - ind(L_n(R))} = p^{n^2 - \left\lfloor \frac{n^2}{4} \right\rfloor}.$$

When $n = 3$, $L_3(R) = W(R)$, so $|L_3(F_p)| = p^{9-2} = p^7$. Finally, the product of two matrices in $L_3(R)$ has the form

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{13}b_{31} & 0 & a_{11}b_{13} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{33}b_{31} & 0 & a_{31}b_{13} + a_{33}b_{33} \end{pmatrix}.$$

We now have a similar case to $T(R)$ but with a more complicated structure.

6. Conclusions and Future Directions

In Section 3, the groundwork is laid to determine the null ideal of $T(D)$ for the specially defined matrix ring $T(D)$ where D is an integral domain. We investigated the set of polynomials in the null ideal $N_{T(D)}(T(D))$ and showed that it can be determined by taking some algebraic manipulations. A consequence is that the null set is a 2-sided ideal of the matrix ring $M_n(D)$ and the structure is described. In section 4, additional properties of the null polynomials are provided. In section 5, we identify some subsets which also form subrings of $M_n(R)$. Such subrings are described by the newly defined zero-indecies of matrices.

Section 5 is intended to give motivation for further exploration into subrings of different forms. The ring $T(R)$ being a subring of $L_3(R)$ does not imply that the null ideal of $T(R)$ is an ideal of $L_3(R)$. In a similar fashion to Lemma 3.2 we construct functions to populate the elements of a matrix with the resulting polynomials after reducing the operations of addition and multiplication for each polynomial in $L_n(R)[x]$ with coefficients in $L_n(R)$.

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