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Ingrid Vukusic

Supervisors:

Home University: Assoz. Prof. Dipl.-Ing. Dr. Volker Ziegler,
Mathematics Department, University of Salzburg, Austria

Host University: Dr. Eva Goedhart,
Mathematics Department, Franklin & Marshall College, Pennsylvania, USA

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During the research stay, three papers were written. Each of them is submitted for publication.

Publication I: I. Vukusic and V. Ziegler. On sums of Fibonacci numbers that are powers of numbers with limited hamming weight. Submitted, 2023. arXiv:2302.08303.

Publication II: B. Earp-Lynch, B. Faye, E. G. Goedhart, I. Vukusic, and D. P. Wisniewski. On a simple quartic family of Thue equations over imaginary quadratic number fields. Submitted, 2023. arXiv:2303.15243.

Publication III: B. Faye, I. Vukusic, E. Waxman, and V. Ziegler. Thue equations over $\mathbb{C}(T)$: The complete solution of a simple quartic family. Submitted, 2023. arXiv:2301.06129.

Note that actually only the first paper is on the topic "On sums of linear recurrence sequences that are perfect powers". As planned, we proved the following: For any fixed k, if y can be written as the sum of k Fibonacci numbers $y = F_{n_1} + \cdots + F_{n_k}$ and if y is large enough, then the equation $F_n + F_m = y^a$ has no solutions with $a \geq 2$. We also considered the general case (which was part of the plan as well), and we are confident that with the same methods we can prove the following: For fixed linear recurrence sequences $(U_n)_{n\in\mathbb{N}}, (V_m)_{m\in\mathbb{N}}$ and integer coefficients $a_1, \dots, a_k, b_1, \dots, b_\ell$, with some mild technical restrictions, the system of equations

$$\begin{aligned} a_1 U_{n_1} + \cdots + a_k U_{n_k} &= y^a, \\ b_1 V_{m_1} + \cdots + b_\ell V_{m_\ell} &= y^b \end{aligned}$$

has no solutions with large y. Writing down the proof is planned for future work. As already mentioned, the methods for the general problem are basically the same as in Publication I (lower bounds for linear forms in logarithms and finite induction). This is the reason why it was more fruitful to join the project of Publication II. It was an open project, where the main outline of the proof had already been worked out by the other coauthors, but a lot of the details were still missing. This was the perfect opportunity to learn the hypergeometric method, which is used in similar settings as lower bounds for linear forms in logarithms, but is rather different. This project was successfully completed. Finally, Publication III was also completed during the research stay. It is concerned with the same family of Thue equations as Publication II, but in the function field setting. Since the paper was already almost finished before the start of the research stay, and related to the second paper, it made sense to finish this paper as well.

ON SUMS OF TWO FIBONACCI NUMBERS THAT ARE POWERS OF NUMBERS WITH LIMITED HAMMING WEIGHT

INGRID VUKUSIC AND VOLKER ZIEGLER

Abstract. In 2018, Luca and Patel conjectured that the largest perfect power representable as the sum of two Fibonacci numbers is $3864^2 = F_{36} + F_{12}$. In other words, they conjectured that the equation

$$(*) \qquad y^a = F_n + F_m$$

has no solutions with $a \geq 2$ and $y^a > 3864^2$. While this is still an open problem, there exist several partial results. For example, recently Kebli, Kihel, Larone and Luca proved an explicit upper bound for y^a, which depends on the size of y.

In this paper, we find an explicit upper bound for y^a, which only depends on the Hamming weight of y with respect to the Zeckendorf representation. More specifically, we prove the following: If $y = F_{n_1} + \cdots + F_{n_k}$ and equation ($*$) is satisfied by y and some non-negative integers n, m and $a \geq 2$, then

$$y^a \leq \exp\left(C(\varepsilon) \cdot k^{(3+\varepsilon)k^2}\right).$$

Here, $\varepsilon > 0$ can be chosen arbitrarily and $C(\varepsilon)$ is an effectively computable constant.

1. Introduction

The Fibonacci numbers, defined by $F_0 = 0$, $F_1 = 1$ and $F_{k+2} = F_{k+1} + F_k$ for $k \geq 0$, might be the most popular linear recurrence sequence of all. They have a great many beautiful properties and a vast amount of research has been done on problems involving Fibonacci numbers. For instance, it was a long-standing conjecture that 0, 1, 8 and 144 are the only Fibonacci Numbers that are perfect powers. This conjecture was proven in 2003 by Bugeaud, Mignotte and Siksek [3]. In view of this result, it was a natural next step to search for all perfect powers that are sums of two Fibonacci numbers, i.e. to try and solve the equation

$$(1) \qquad F_n + F_m = y^a,$$

where n, m, y, a are non-negative integers with $a \geq 2$. There are 18 solutions known with $n \geq m \geq 0$, the largest being $F_{36} + F_{12} = 3864^2$. In 2018, Luca and Patel [9] conjectured that these are the only solutions to equation (1). They proved their conjecture in the case that $n \equiv m \pmod 2$. The general conjecture, however, remains open.

Let us summarize further existing partial results on this conjecture. If $m = 0$, then we have $F_n = y^a$, which, as mentioned above, was solved in [3]. For $m = 1, 2$ we have the equation $F_n + 1 = y^a$, which was solved by Bugeaud, Luca, Mignotte, Siksek in 2006 [2]. For any fixed y it is in principal possible to solve equation (1) completely. For example, Bravo and Luca [1] solved the equation $F_n + F_m = 2^a$. In the general case with fixed y, explicit upper bounds for n, m and a in terms of y were established recently in [5] and in [6]. Moreover, Kebli, Kihel, Larone and Luca [5] proved that the abc-conjecture implies that (1) has only finitely many solutions. Most recently, Ziegler [11] proved that for any fixed y equation (1) has at most one solution with $a \geq 1$, unless $y = 2, 3, 4, 6, 10$. In particular,

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his result implies that if y can be represented as $y = F_{n_1} + F_{n_2}$, then equation (1) has no solutions with $a \geq 2$ (with the exceptions $y = 2, 3, 4, 6, 10$).

In this paper, we want to make another step towards solving equation (1), and generalize the above mentioned results in the following way: Instead of fixing y or requiring that it have the form $y = F_{n_1} + F_{n_2}$, we allow arbitrary y with bounded Hamming weight with respect to the Zeckendorf representation (i.e. $y = F_{n_1} + \cdots + F_{n_k}$ with bounded k). More specifically, we give an explicit upper bound for any perfect power y^a that is a sum of two Fibonacci numbers $y^a = F_n + F_m$, and the upper bound does not depend on the size of y, but only on the Hamming weight of the Zeckendorf representation of y. We now state our main result.

Theorem 1. *Let $\varepsilon > 0$. Then there exists an effectively computable constant $C(\varepsilon)$ such that the following holds. If the equations*

$$y = F_{n_1} + \cdots + F_{n_k} \quad \text{and} \tag{A}$$

$$y^a = F_n + F_m \tag{B}$$

are satisfied by some non-negative integers $y, a, n_1, \ldots, n_k, n, m$ with $a \geq 2$, then

$$y^a \leq \exp\left(C(\varepsilon) \cdot k^{(3+\varepsilon)k^2}\right). \tag{2}$$

Remark 1. The constant $C(\varepsilon)$ can indeed be computed from our proof. However, it will be extremely large and not useful in practice. This is because we chose to write the upper bound in a way that is both simple and asymptotically good. So if one chooses an $\varepsilon > 0$ and computes the constant $C(\varepsilon)$ such that (2) holds for all k, the bound will be extremely bad for small k.

For computing an actual upper bound for a given k, we recommend going to equation (16) in Section 6 and computing the maximum of the expressions T_{k+1} over all $1 \leq \ell_0 \leq k$. Then, one can proceed as described in Remark 2 in Section 2.1 and solve the inequality $n < 6 \cdot 10^{29} \cdot T_{k+1}^4$.

Let us outline the rest of the paper and the strategy of our proof. In Section 2, we state some preliminary results related to Fibonacci numbers and our problem, as well as lower bounds for linear forms in logarithms and an inequality. In Section 3, we construct a total of $2k$ "basic" linear forms in logarithms from equations (A) and (B). Each of these linear forms will contain the unknown logarithm $\log y$. The main idea of the proof is the following: In several steps (k or $k+1$ steps), we take two of the "basic" linear forms at a time and eliminate $\log y$. Then we apply lower bounds for linear forms in logarithms to the new linear form and obtain an upper bound for one of the expressions $n - m, n_1 - n_2, \ldots, n_1 - n_k, n_1$. Depending on how large $n - m$ is compared to $n_1 - n_2, \ldots, n_1 - n_k, n_1$, we need to do a slightly different succession of steps. An overview of these steps can be found in Figure 1 in Section 4. The exact bounds, that are obtained in each step, are computed in Section 5. We "walk the steps" in Section 6: Depending on which path of steps we walk, we end up with a different bound for n_1 in the last step, see Figure 2. In Section 6, we compute these bounds and find a common upper bound for n_1 of the shape $n_1 < c \cdot (\log n)^x$. In particular, this implies $\log y < c \cdot (\log n)^x$. Finally, in Section 7 we combine this bound with the bound from [5], which is of the shape $n < c(\log y)^4$. Thus, we end up with an inequality of the shape $n < c \cdot (\log n)^x$, which implies an upper bound for n (see Remark 2). Let us moreover point out that we use the result from [9] to exclude the case $n \equiv m \pmod 2$. This is very helpful for checking that our linear forms in logarithms don't vanish.

Of course, the strategy of applying lower bounds for linear forms in logarithms to exponential Diophantine equations has been well known and extensively used for a long time. In this paper, we use two particular tricks: the elimination of unknown logarithms and a kind of finite induction. Both of these tricks have been used in several papers before (see e.g. [8] for the elimination of unknown logarithms and [7] for the finite induction method), however, to the authors' best knowledge, this is the first time that they are used in a combined way.

Moreover, the induction is not just a straightforward induction over k steps, but different cases lead to quite different bounds. It is interesting to see how the bounds depend on the cases, and to then determine an overall asymptotically good bound.

2. Preliminary results

In this section we, start by recalling some basic properties of Fibonacci numbers. Moreover, we argue why we may assume $n_k \geq 2$ and $n_{i+1} \geq n_i + 2$ for $i = 1, \ldots, k-1$, as well as $n - 2 \geq m \geq 2$ in the rest of the paper. Then we state some known results and some elementary results related to equations (A) and (B).

In the second subsection, we state one of Matveev's lower bounds for linear forms in logarithms. Furthermore, we prove an elementary lemma that will allow us to deduce an absolute bound for n from a bound of the shape $n \leq c \cdot (\log n)^x$.

2.1. Results related to Fibonacci numbers

For the Fibonacci numbers we have the well known Binet formula

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \quad \text{where} \quad \alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = -1/\alpha = \frac{1-\sqrt{5}}{2}.$$

In Theorem 1 the representations $F_{n_1} + \cdots + F_{n_k}$ and $F_n + F_m$ are not necessarily Zeckendorf representations, i.e. we might have consecutive or identical indices, or indices equal to 0 or 1. However, in the rest of this paper we will assume that $n_k \geq 2$ and $n_{i+1} \geq n_i + 2$ for $i = 1, \ldots, k-1$, as well as $n - 2 \geq m \geq 2$. Let us justify now why we can do this. If $F_{n_1} + \cdots + F_{n_k}$ is not a Zeckendorf representation, then we can consider the Zeckendorf representation $F_{n'_1} + \cdots + F_{n'_{k'}}$ instead. Since the Zeckendorf representation is minimal (see e.g. [4, Theorem 1.1]) we have $k' \leq k$ and we will get the same or an even stronger result. If $F_n + F_m$ is not a Zeckendorf representation, then either $n = m$, or the Zeckendorf representation in fact only consists of one Fibonacci number and we have $y^a = F_{n'}$. As mentioned in the Introduction, the latter is by [3] only possible for $y^a \leq 144$, in which case we are done. If $n = m$, then $y^a = 2F_n$ implies $F_n = 2^s (y')^a$, for suitable y' and s. From [2, Theorem 4] it follows that this is only possible for $n = 1, 2, 3, 6, 12$ (cf. [9, Theorem 2]). In fact, $y^a = 2F_n$ only works for $n = 3, 6$ and thus $y^a \leq 2F_6 = 16$, and we are done as well.

Finally, note that we may assume $y \geq 2$, since Theorem 1 is trivial for $y = 0, 1$.

Next, we state the result due to Luca and Patel [9], that we mentioned in the Introduction. We will use it to exclude the case $n \equiv m \pmod 2$.

Theorem A (Luca and Patel, 2018)**.** *Let (y, a, n, m) be a solution to (B), i.e.*

$$y^a = F_n + F_m,$$

with $y \geq 1$, $a \geq 2$ and $n - 2 \geq m \geq 2$. If $n \equiv m \pmod 2$, then $n \leq 36$.

Given integers $y, a, n > m$ that satisfy equation (B), one can use lower bounds for linear forms in logarithms to obtain a bound for n in terms of y. This was done explicitly by Kebli et al. [5, Theorem 1] and we will use their bound in our proof.

Theorem B (Kebli et al., 2021)**.** *Let (y, a, n, m) be a solution to (B), i.e.*

$$y^a = F_n + F_m,$$

with $y \geq 2$, $a \geq 2$ and $n \geq m \geq 0$. Then

$$a < n < 6 \cdot 10^{29} (\log y)^4.$$

Next, we consider equation (A) and observe the following.

Lemma 1. *Let $(y, n_1, \ldots, n_k)$ be a solution to (A), i.e.*

$$y = F_{n_1} + \cdots + F_{n_k},$$

with $n_k \geq 2$ and $n_{i+1} \geq n_i + 2$ for $i = 1, \ldots, k-1$. Then

$$\log y < n_1.$$

Proof. By the properties of the Zeckendorf expansion, we have $y = F_{n_1} + \cdots + F_{n_k} < 2F_{n_1}$, which implies $\log y < \log(2F_{n_1}) = \log 2 + \log(\alpha^{n_1} - \beta^{n_1}) - \log\sqrt{5} < n_1$. □

Remark 2. In view of Theorem B and Lemma 1, we are done if we manage to prove a bound for n_1 of the shape $n_1 \ll (\log n)^x$: Then we have $n < 6\cdot 10^{29}(\log y)^4 \le 6\cdot 10^{29} n_1^4 \ll (\log n)^{4x}$, which immediately gives us an effective upper bound for n. This will indeed be our strategy.

We will use the following not very sharp but simple estimates. Note that there is no benefit from making them sharper, as the constants coming from these estimates will be "swallowed" by a much larger constant in Section 5.

Lemma 2. *For any $k \in \mathbb{Z}_{\ge 1}$ and $n_1, \dots, n_k \in \mathbb{Z}$ with $n_1 > \cdots > n_k \ge 0$, we have*

$$|\alpha^{-n_1} + \cdots + \alpha^{-n_k}| < 3 \quad \text{and} \tag{3}$$

$$|\beta^{n_1} + \cdots + \beta^{n_k}| < 3. \tag{4}$$

Proof. Inequality (3) follows from a simple estimation:

$$0 \le \alpha^{-n_1} + \cdots + \alpha^{-n_k} \le \sum_{i=0}^{\infty} \alpha^{-i} = \frac{1}{1-\alpha^{-1}} < 3.$$

Then (4) follows easily as well:

$$|\beta^{n_1} + \cdots + \beta^{n_k}| \le |\beta|^{n_1} + \cdots + |\beta|^{n_k} = \alpha^{-n_1} + \cdots + \alpha^{-n_k} < 3.$$

□

The next lemma will be used to bound the coefficients in the linear forms in Section 5. For simplicity, the estimates are very rough. Sharper estimates would improve the bound (17) at the end of Section 6 only marginally.

Lemma 3. *Let $(y, a, n_1, \dots, n_k, n, m)$ be a solution to (A) and (B) with $y, a, n_k \ge 2$ and $n_{i+1} \ge n_i + 2$ for $i = 1, \dots, k-1$, as well as $n - 2 \ge m \ge 2$. Then we have*

$$n_1 < n \quad \text{and} \quad a < n.$$

Proof. Since $y^a > y$, it follows immediately from the properties of the Zeckendorf representation, that $n_1 < n$.

For the second inequality note that we have $2^a \le y^a = F_n + F_m < 2F_n < 2\alpha^n$, which implies $a < \log 2 + n \cdot \log\alpha / \log 2 < n$ for $n \ge m + 2 \ge 4$.

□

2.2. Result related to the application of lower bounds for linear forms in logarithms

Our proof will heavily rely on lower bounds for linear forms in logarithms. In order to switch from expressions of the shape $|\eta_1^{b_1} \cdots \eta_t^{b_t} - 1|$ to linear forms in logarithms of the shape $|b_1 \log\eta_1 + \cdots + b_t \log\eta_t|$, we will use the following easy-to-check lemma.

Lemma 4. *If $|x - 1| \le 0.5$, then $|\log x| \le 2|x-1|$.*

In order to compute lower bounds for expressions of the shape $|b_1 \log\eta_1 + \cdots + b_t \log\eta_t|$, we will use Matveev's popular result [10, Corollary 2.3] because it is very good and easy to apply.

Let us first recall the definition of the height and some basic properties. Let η be an algebraic number of degree d over the rationals, with minimal polynomial

$$a_0(X - \eta^{(1)}) \cdots (X - \eta^{(d)}) \in \mathbb{Z}[X].$$

Then the absolute logarithmic height of η is given by

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\max\{1, |\eta^{(i)}|\}\right).$$

For any algebraic numbers $\eta_1, \ldots, \eta_t \in \overline{\mathbb{Q}}$ and $z \in \mathbb{Z}$, the following well known properties hold:

$$h(\eta_1 + \cdots + \eta_t) \leq h(\eta_1) + \cdots + h(\eta_t) + \log t,$$
$$h(\eta^z) = |z| \cdot h(\eta).$$

Theorem C (Matveev, 2000)**.** *Let $\eta_1, \ldots, \eta_t$ be positive real algebraic numbers in a number field K of degree D, let $b_1, \ldots b_t$ be rational integers and assume that*

$$\Lambda := b_1 \log \eta_1 + \cdots + b_t \log \eta_t \neq 0.$$

Then

$$\log |\Lambda| \geq -1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_t,$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$
$$A_i \geq \max\{Dh(\eta_i), |\log \eta_i|, 0.16\} \quad \textit{for all } i = 1, \ldots, t.$$

In order to be able to apply Theorem C, one has to check that the linear form Λ does not vanish. This often requires some tricks. In Section 5 we will make use of the following lemma.

Lemma 5. *Let x be an odd positive integer. Then $\alpha^x + 1$ is not divisible by $\sqrt{5}$ (in the principal ideal domain $\mathbb{Z}[\alpha]$, where $\alpha = \frac{1+\sqrt{5}}{2}$).*

Proof. Let x be an odd positive integer. If $\alpha^x + 1$ were divisible by $\sqrt{5}$, then the norm $N_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}(\alpha^x + 1)$ would be divisible by 5. Let us compute the norm:

$$N_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}(\alpha^x + 1) = (\alpha^x + 1)(\beta^x + 1) = (\alpha\beta)^x + (\alpha^x + \beta^x) + 1 = (-1)^x + L_x + 1 = L_x,$$

where L_x is the x-th Lucas number (the Lucas numbers are defined by $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$, and they indeed have the Binet representation $L_n = \alpha^n + \beta^n$). Modulo 5 the Lucas sequence looks like this: $2, 1, 3, 4, 2, 1, \ldots$ and in particular no Lucas number is divisible by 5. Thus $N_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}(\alpha^x + 1)$ is not divisible by 5 for odd x, and $\alpha^x + 1$ cannot be divisible by $\sqrt{5}$. □

Finally, after the repeated application of lower bounds for linear forms in logarithms, we will end up with an inequality of the shape $n \leq c \cdot (\log n)^x$ by the end of Section 6. In order to obtain an absolute upper bound for n, we will use the following lemma in Section 7.

Lemma 6. *Let $\delta > 0$ and let $n, c, x \in \mathbb{R}_{\geq 1}$ satisfy the inequality*

$$n \leq c \cdot (\log n)^x. \tag{5}$$

Then

$$n \leq \max\left\{\exp(\exp((1 + \delta^{-1})^2)),\ 2^x \cdot c \cdot (\log c)^x,\ (2x)^{(1+\delta)x} \cdot c\right\}.$$

Proof. Let n, c, x be numbers ≥ 1 that satisfy (5).

First, note that if $n \leq e^e$, then we have $n < \exp(\exp((1 + \delta^{-1})^2))$ and we are done immediately. Therefore, we may assume that $n > e^e$, which implies

$$\log \log \log n > 0. \tag{6}$$

Next, assume for a moment that $\delta/(1 + \delta) \cdot \log \log n \leq \log \log \log n$. Taking logarithms, we obtain

$$\log(\delta/(1 + \delta)) + \log \log \log n \leq \log \log \log \log n.$$

Since $\log y < y/2$ for any $y > 0$, and $\log \log \log n > 0$ by (6), we obtain

$$\log(\delta/(1 + \delta)) + \log \log \log n \leq \log \log \log \log n < (\log \log \log n)/2,$$

which implies

$$\log \log \log n < 2 \log((1 + \delta)/\delta) = 2 \log(1 + \delta^{-1}).$$

But this immediately implies $n < \exp(\exp((1+\delta^{-1})^2))$ and we are done. Therefore, we may from now on assume that

$$\log\log\log n < \delta/(1+\delta) \cdot \log\log n. \tag{7}$$

Now we consider inequality (5) and take logarithms, obtaining

$$\log n \leq \log c + x \log\log n. \tag{8}$$

Assume for the moment that $x \log\log n \leq (\log n)/2$. Then the above inequality implies

$$\log n \leq 2 \log c,$$

which plugging back into (5) immediately yields $n \leq 2^x \cdot c \cdot (\log c)^x$ and we are done as well.

Finally, we consider the case that $x \log\log n > (\log n)/2$, i.e. $\log n < 2x \log\log n$. Taking logarithms, we obtain

$$\log\log n < \log(2x) + \log\log\log n.$$

Together with (7) this yields

$$\log\log n < \log(2x) + \delta/(1+\delta) \cdot \log\log n,$$

which implies

$$\left(1 - \frac{\delta}{1+\delta}\right)\log\log n = \frac{1}{1+\delta} \cdot \log\log n < \log(2x),$$

and then

$$\log\log n < (1+\delta)\log(2x).$$

Plugging this into (8), we obtain

$$\log n < \log c + x \cdot (1+\delta)\log(2x),$$

which implies

$$n < c \cdot (2x)^{(1+\delta)x}.$$

□

3. Constructing the basic linear forms in logarithms

In this section, we construct k linear forms in logarithms from equation (A) and k linear forms from equation (B). In the third subsection we give an overview of all the linear forms in logarithms and their upper bounds.

3.1. Linear forms coming from equation (A)

Using the Binet formula, we can rewrite equation (A) as

$$y = \frac{\alpha^{n_1} - \beta^{n_1}}{\sqrt{5}} + \cdots + \frac{\alpha^{n_k} - \beta^{n_k}}{\sqrt{5}}.$$

Multiplication by $\sqrt{5}$ yields

$$y\sqrt{5} = \alpha^{n_1} + \cdots + \alpha^{n_k} - (\beta^{n_1} + \cdots + \beta^{n_k}). \tag{9}$$

To obtain the first linear form in logarithms, we shift the largest power α^{n_1} to the left hand side, take absolute values and use Lemma 2:

$$\begin{aligned} |y\sqrt{5} - \alpha^{n_1}| &= |\alpha^{n_2} + \cdots + \alpha^{n_k} - (\beta^{n_1} + \cdots + \beta^{n_k})| \\ &\leq |\alpha^{n_2}(\alpha^{n_3-n_2} + \cdots + \alpha^{n_k-n_2})| + |\beta^{n_1} + \cdots + \beta^{n_k}| \\ &\leq \alpha^{n_2} \cdot 3 + 3 \\ &\leq 6\alpha^{n_2}. \end{aligned}$$

Now we divide by α^{n_1} and obtain

$$\left|\frac{y\sqrt{5}}{\alpha^{n_1}} - 1\right| \leq 6\alpha^{-(n_1-n_2)}.$$

If $n_1 - n_2 \geq 6$, then the above expressions are ≤ 0.5, so with Lemma 4 we get

$$|\log y + \sqrt{5} - n_1 \log \alpha| \leq 12\alpha^{-(n_1-n_2)}.$$

This is our first linear form in logarithms. Next, we construct more of them by shifting more powers of α to the left hand side.

Let us go back to (9) and shift the largest ℓ powers $\alpha^{n_1}, \ldots, \alpha^{n_\ell}$ to the left ($2 \leq \ell \leq k-1$). Then, as above, we obtain

$$|y\sqrt{5} - (\alpha^{n_1} + \cdots + \alpha^{n_\ell})| = |\alpha^{n_{\ell+1}} + \cdots + \alpha^{n_k} - (\beta^{n_1} + \cdots + \beta^{n_k})| \leq 6\alpha^{n_{\ell+1}}.$$

Dividing by $\alpha^{n_1} + \cdots + \alpha^{n_\ell} = \alpha^{n_1}(1 + \alpha^{n_2-n_1} + \cdots + \alpha^{n_\ell-n_1}) \geq \alpha^{n_1}$ we obtain

$$\left| \frac{y\sqrt{5}}{\alpha^{n_1}(1 + \alpha^{n_2-n_1} + \cdots + \alpha^{n_\ell-n_1})} \right| \leq 6\alpha^{-(n_1-n_{\ell+1})}.$$

Then, if $n_1 - n_{\ell+1} \geq 6$, by Lemma 4 we have

$$|\log y + \log\sqrt{5} - n_1 \log \alpha - \log(1 + \alpha^{n_2-n_1} + \cdots + \alpha^{n_\ell-n_1})| \leq 12\alpha^{-(n_1-n_{\ell+1})}.$$

Finally, let us construct the last linear form in logarithms by shifting all powers of α to the left hand side. Then from equation (9) we obtain

$$|y\sqrt{5} - (\alpha^{n_1} + \cdots + \alpha^{n_k})| = |\beta^{n_1} + \cdots + \beta^{n_k}| \leq 3 \leq 6.$$

Dividing by $\alpha^{n_1} + \cdots + \alpha^{n_k} = \alpha^{n_1}(1 + \alpha^{n_2-n_1} + \cdots + \alpha^{n_k-n_1}) \geq \alpha^{n_1}$, we obtain

$$\left| \frac{y\sqrt{5}}{\alpha^{n_1}(1 + \alpha^{n_2-n_1} + \cdots + \alpha^{n_k-n_1})} \right| \leq 6\alpha^{-n_1}.$$

Then, as above, if $n_1 \geq 6$, with Lemma 4 we obtain

$$|\log y + \log\sqrt{5} - n_1 \log \alpha - \log(1 + \alpha^{n_2-n_1} + \cdots + \alpha^{n_k-n_1})| \leq 12\alpha^{-n_1}.$$

3.2. Linear forms coming from Equation (B)

We rewrite equation (B) using the Binet formula:

$$y^a = \frac{\alpha^n - \beta^n}{\sqrt{5}} + \frac{\alpha^m - \beta^m}{\sqrt{5}}.$$

Then, by the same procedure as above, if $n - m \geq 6$ and $n \geq 6$, we obtain the two linear forms

$$|a \log y + \sqrt{5} - n \log \alpha| \leq 12\alpha^{-(n-m)} \quad \text{and}$$
$$|a \log y + \log\sqrt{5} - m \log \alpha - \log(\alpha^{n-m} + 1)| \leq 12\alpha^{-n}.$$

3.3. Overview of basic linear forms

Let us sum up all the linear forms in logarithms that we have constructed and name them in an appropriate way: Let us denote the k linear forms coming from equation (A) by $\Lambda_{A1}, \ldots, \Lambda_{Ak}$, and the two linear forms coming from equation (B) by $\Lambda_{B1}, \Lambda_{B2}$. Specifically, for $2 \leq l \leq k-1$, we have

$$\begin{aligned}
|\Lambda_{A1}| &:= |\log y + \log\sqrt{5} - n_1 \log \alpha| \leq 12\alpha^{-(n_1-n_2)}, \\
|\Lambda_{A\ell}| &:= |\log y + \log\sqrt{5} - n_1 \log \alpha - \log(1 + \alpha^{n_2-n_1} + \cdots + \alpha^{n_\ell-n_1})| \leq 12\alpha^{-(n_1-n_{\ell+1})}, \\
|\Lambda_{Ak}| &:= |\log y + \log\sqrt{5} - n_1 \log \alpha - \log(1 + \alpha^{n_2-n_1} + \cdots + \alpha^{n_k-n_1})| \leq 12\alpha^{-n_1}, \\
|\Lambda_{B1}| &:= |a \log y + \log\sqrt{5} - n \log \alpha| \leq 12\alpha^{-(n-m)}, \\
|\Lambda_{B2}| &:= |a \log y + \log\sqrt{5} - m \log \alpha - \log(\alpha^{n-m} + 1)| \leq 12\alpha^{-n}.
\end{aligned}$$

The upper bounds are all of the shape $12\alpha^{-X}$. Mind that each of the inequalities only holds if $X \geq 6$. However, the goal will always be to bound X from above, so if $X \leq 6$, we will just skip the corresponding step.

4. Eliminating y from the linear forms and overview of the steps

In each of the linear forms $\Lambda_{A1}, \dots, \Lambda_{Ak}, \Lambda_{B1}, \Lambda_{B2}$ we have one logarithm that is unknown, namely $\log y$. Therefore, we will always take two distinct linear forms and eliminate $\log y$. For example, if we take Λ_{A1} and Λ_{B1} and eliminate $\log y$, we get a new linear form Λ_{A1}^* in fixed logarithms with a bound of the shape $|\Lambda_{A1}^*| \leq a \cdot 12\alpha^{-(n_1-n_2)} + 12\alpha^{-(n-m)}$. After computing a lower bound for $|\Lambda_{A1}^*|$ with Matveev's theorem, we then obtain an upper bound either for $n_1 - n_2$ or for $n - m$, depending on which one is smaller. This leads to different cases and in each case we have to continue with slightly different steps. Figure 1 shows an overview of the steps. Each rectangular box stands for a step, where we take the two linear forms written in the box and eliminate y. The obtained upper bound is written along the arrow that points to the next step. In the steps on the left (Steps A1 to Ak), there are always two cases, depending on whether $n_1 - n_\ell$ (or n_1) or $n - m$ is smaller. If we ever cross over to the steps on the right (Steps B1 to Bk), then we just follow the arrows pointing downwards, always obtaining bounds for $n_1 - n_\ell$ (or n_1). The purpose of Figure 1 is to give a rough idea of the proof. There will be a more detailed figure in Section 6. We now construct and estimate all the linear forms in logarithms that are used in the steps.

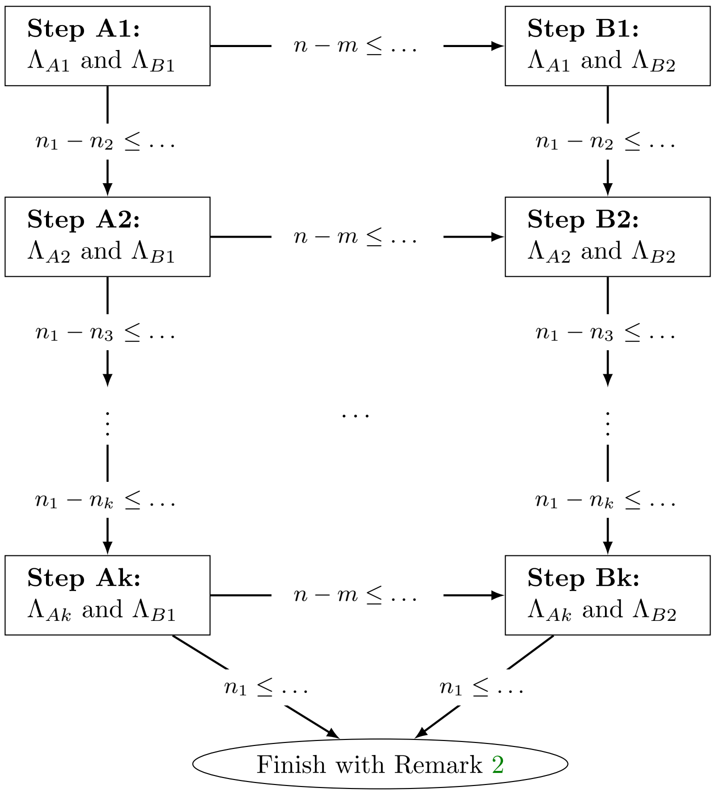

FIGURE 1. Overview of steps

Let us start with the steps on the left (Step A1 to Step Ak). In Step A1 we eliminate $\log y$ by computing the new linear form $\Lambda_{A1}^* := a\Lambda_{A1} - \Lambda_{B1}$:

$$\begin{aligned} |\Lambda_{A1}^*| = |a\Lambda_{A1} - \Lambda_{B1}| &= |(a-1)\log\sqrt{5} + (n - an_1)\log\alpha| \\ &\leq a \cdot 12\alpha^{-(n_1-n_2)} + 12\alpha^{-(n-m)} \leq 18a\alpha^{-\min\{n_1-n_2,n-m\}}, \end{aligned}$$

where we used $a \geq 2$. In the same way we also obtain the linear forms for Steps A2 to Ak:

$$\begin{aligned}|\Lambda^*_{A\ell}| &:= |a\Lambda_{A\ell} - \Lambda_{B1}| = |(a-1)\log\sqrt{5} + (n - an_1)\log\alpha - a\log(1+\alpha^{n_2-n_1}+\cdots+\alpha^{n_\ell-n_1})| \\ &\leq 18a\alpha^{-\min\{n_1-n_{\ell+1},n-m\}} \quad \text{for } l = 2,\ldots,k-1, \\ |\Lambda^*_{Ak}| &:= |a\Lambda_{Ak} - \Lambda_{B1}| = |(a-1)\log\sqrt{5} + (n - an_1)\log\alpha - a\log(1+\alpha^{n_2-n_1}+\cdots+\alpha^{n_k-n_1})| \\ &\leq 18a\alpha^{-\min\{n_1,n-m\}}.\end{aligned}$$

Analogously, we construct and estimate the linear forms for Steps B1 to Bk. Note that $n_1 - n_\ell < n_1 < n$, so clearly $\alpha^{-n} < \alpha^{-(n_1-n_\ell)}$ for any ℓ and we don't need to write any minima.

$$\begin{aligned}|\Lambda^*_{B1}| &:= |a\Lambda_{A1} - \Lambda_{B2}| = |(a-1)\log\sqrt{5} + (m-an_1)\log\alpha + \log(\alpha^{n-m}+1)| \\ &\leq a\cdot 12\alpha^{-(n_1-n_2)} + 12\alpha^{-n} \leq 18a\alpha^{-(n_1-n_2)}, \\ |\Lambda^*_{B\ell}| &:= |a\Lambda_{A\ell} - \Lambda_{B2}| = |(a-1)\log\sqrt{5} + (m-an_1)\log\alpha \\ &\qquad - a\log(1+\alpha^{n_2-n_1}+\cdots+\alpha^{n_\ell-n_1}) + \log(\alpha^{n-m}+1)| \\ &\leq 18a\alpha^{-(n_1-n_{\ell+1})}, \quad \text{for } l = 2,\ldots,k-1, \\ |\Lambda^*_{Bk}| &:= |a\Lambda_{Ak} - \Lambda_{B2}| = |(a-1)\log\sqrt{5} + (m-an_1)\log\alpha \\ &\qquad - a\log(1+\alpha^{n_2-n_1}+\cdots+\alpha^{n_k-n_1}) + \log(\alpha^{n-m}+1)| \\ &\leq 18a\alpha^{-n_1}. \qquad (10)\end{aligned}$$

Mind that each of these estimates only holds if we have $X \geq 6$ for the exponent in the upper bound $18a\alpha^{-X}$.

5. Application of Matveev's theorem

In this section we apply Matveev's theorem to all the linear forms $\Lambda^*_{A\ell}, \Lambda^*_{B\ell}$, that we obtained in the previous section. Moreover, we compare the lower bounds to the upper bounds and compute general bounds for the exponents $n_1 - n_\ell$, n_1 or $n - m$.

First, we check that the linear forms $\Lambda^*_{A1},\ldots,\Lambda^*_{Ak},\Lambda^*_{B1},\ldots,\Lambda^*_{Bk}$ are non-zero. Note that in each linear form we have the expression $(a-1)\log\sqrt{5}$ and $a-1 \neq 0$. Thus, if a linear form were zero, $\log\sqrt{5}$ would have to be canceled out by the other logarithms. In particular, since $\sqrt{5}$ is prime in $\mathbb{Z}[\alpha]$, it would have to divide at least one argument of a logarithm. We only have three other types of logarithms: $\log\alpha$, $\log(\alpha^{n-m}+1)$ and $\log(1+\alpha^{n_2-n_1}+\cdots+\alpha^{n_\ell-n_1})$. First, since α is a unit in $\mathbb{Z}[\alpha]$, it is not divisible by $\sqrt{5}$, so $\log\alpha$ does not contribute to the cancellation of $\log\sqrt{5}$. Second, by Lemma 5, the expression $\alpha^{n-m}+1$ is not divisible by $\sqrt{5}$ unless $n-m$ is even. If $n-m$ is even, we are immediately done with Theorem A, so let us assume that $n-m$ is odd. Thus $\log(\alpha^{n-m}+1)$ does not contribute to the cancellation of $\log\sqrt{5}$ either. And third, the expression $(1+\alpha^{n_2-n_1}+\cdots+\alpha^{n_\ell-n_1})$ might be divisible by $\sqrt{5}$. However, the coefficient of $\log(1+\alpha^{n_2-n_1}+\cdots+\alpha^{n_\ell-n_1})$ is always $-a$. So if $\log\sqrt{5}$ canceled out completely, we would need to have $(a-1) - a\cdot v_{\sqrt{5}}(1+\alpha^{n_2-n_1}+\cdots+\alpha^{n_\ell-n_1}) = 0$, which is clearly impossible. Here $v_{\sqrt{5}}(\cdot)$ denotes the valuation on $\mathbb{Q}(\sqrt{5})$ associated with the prime ideal $(\sqrt{5})$, normalized by $v_{\sqrt{5}}(\sqrt{5}) = 1$.

Therefore, all the linear forms $\Lambda^*_{A1},\ldots,\Lambda^*_{Ak},\Lambda^*_{B1},\ldots,\Lambda^*_{Bk}$ are non-zero and we can apply Matveev's theorem to each of them. Each linear form has an upper bound of the shape $18a\alpha^{-X}$ and in each step we compare the lower and the upper bound to obtain a bound for the expression X.

We start by describing the last step, because the lower bound for the Λ^*_{Bk} will be the weakest, so for simplicity we will be able to reuse it in the other steps. Of course, one can obtain sharper bounds by considering each linear form separately, in particular for the linear form Λ^*_{A1} in only two logarithms.

5.1. Step Bk

Assume that we already have an upper bound T_k for $n_1 - n_k$ and an upper bound S for $n - m$. We want to apply Matveev's theorem (Theorem C) to Λ^*_{Bk} with $t = 4$ and

$$\begin{aligned} \eta_1 &= \sqrt{5}, & b_1 &= a - 1,\\ \eta_2 &= \alpha, & b_2 &= m - an_1,\\ \eta_3 &= 1 + \alpha^{n_2-n_1} + \cdots + \alpha^{n_k-n_1}, & b_3 &= -a,\\ \eta_4 &= \alpha^{n-m} + 1, & b_4 &= 1. \end{aligned}$$

The four numbers $\eta_1, \eta_2, \eta_3, \eta_4$ are real, positive and belong to $K = \mathbb{Q}(\sqrt{5})$, so we can take $D = 2$. As stated in Lemma 3, we have $a \le n$ and $n_1 \le n$, so $\max\{|b_1|, |b_2|, |b_3|, |b_4|\} \le n^2$ and we can set $B = n^2$. Since $h(\sqrt{5}) = (\log 5)/2$ and $h(\alpha) = (\log\alpha)/2$, we can set $A_1 = \log 5$ and $A_2 = \log\alpha$. Finally, we estimate (rather roughly) the heights of η_3 and η_4. Recall that we are assuming $n_1 - n_k \le T_k$ and $n - m \le S$.

$$\begin{aligned} h(1 + \alpha^{n_2-n_1} + \cdots + \alpha^{n_k-n_1}) &\le h(1) + h(\alpha^{n_2-n_1}) + \cdots + h(\alpha^{n_k-n_1}) + \log k\\ &= 0 + (n_1 - n_2)h(\alpha) + \cdots + (n_1 - n_k)h(\alpha) + \log k\\ &\le (k-1)(n_1 - n_k)\frac{\log\alpha}{2} + \log k\\ &\le kT_k.\\ h(\alpha^{n-m} + 1) &\le (n-m)\frac{\log\alpha}{2} + \log 2 \le S. \end{aligned}$$

Thus we can set $A_3 = 2kT_k$ and $A_4 = 2S$. Now Matveev's theorem (Theorem C) tells us that

$$\log|\Lambda^*_{Bk}| \ge -1.4 \cdot 30^{4+3} \cdot 4^{4.5} \cdot 2^2(1 + \log 2)(1 + \log(n^2)) \log 5 \cdot \log\alpha \cdot 2kT_k \cdot 2S.$$

We can estimate $1 + \log n^2 = 1 + 2\log n \le 3\log n$ (this estimate holds for $n \ge 3$; if $n < 3$, we are immediately done). Then we simplify the above lower bound to

$$\log|\Lambda^*_{Bk}| \ge -C \cdot \log n \cdot \log\alpha \cdot k \cdot T_k \cdot S, \tag{11}$$

with

$$\begin{aligned} C &:= 2.1 \cdot 10^{15}\\ &\ge 1.4 \cdot 30^{4+3} \cdot 4^{4.5} \cdot 2^2(1 + \log 2) \cdot 3 \cdot \log 5 \cdot 2 \cdot 2. \end{aligned}$$

Together with (10) this yields

$$-C \cdot \log\alpha \cdot S \cdot T_k \cdot \log n \le \log|\Lambda^*_{Bk}| \le \log 18 + \log a - n_1 \log\alpha,$$

which implies

$$n_1 \le CkST_k \log n =: T_{k+1}. \tag{12}$$

Note that we are allowed to omit the expressions $\log 18$ and $\log a$ because $\log a \le \log n$ and the constant C is extremely large and was estimated very roughly. Moreover, note that the upper bound coming from (10) only holds if $n \ge 6$. However, if $n < 6$, then (12) is trivially fulfilled. An analogous argument will implicitly be used in all other steps as well.

5.2. Steps B1 to Bk–1

First, assume that $\ell \in \{2, \ldots, k-1\}$ and assume that we have already found bounds $T_\ell \ge n_1 - n_\ell$ and $S \ge n - m$. Then in Step Bℓ we consider the linear form $\Lambda^*_{B\ell}$ and, completely analogously to Step Bk, we obtain a lower bound $\log|\Lambda^*_{B\ell}| \ge -C \cdot \log n \cdot \log\alpha \cdot \ell \cdot T_\ell \cdot S$. By comparing the upper and lower bound for $|\Lambda^*_{B\ell}|$ we then, analogously to Step Bk, obtain

$$n_1 - n_{\ell+1} \le C\ell ST_\ell \log n =: T_{\ell+1}. \tag{13}$$

Now let us have a look at Step B1. The linear form Λ^*_{B1} has only three logarithms (instead of four), so it is possible to get a better lower bound. However, for simplicity, we can set

$T_1 = 1$ and see that a bound analogous to (11) still holds: $\log|\Lambda^*_{B1}| \geq -C\cdot\log n\cdot\log\alpha\cdot 1\cdot 1\cdot S$. Thus, in Step B1 we can also obtain the bound (13) for $\ell = 1$ and $T_1 = 1$.

5.3. Steps A1 to Ak

Let $\ell \in \{1,\dots,k\}$. In Step Aℓ we assume that we already have a bound $R_\ell \geq n_1 - n_\ell$ (if $\ell = 1$, we have no bound yet). We consider the linear form $\Lambda^*_{A\ell}$, which is almost exactly equal to the corresponding linear form $\Lambda^*_{B\ell}$, except that the logarithm $\log(\alpha^{n-m}+1)$ is missing. Thus, we only have three logarithms (or even two logarithms if $\ell = 1$) and we can obtain even better lower bounds than in (11). However, for simplicity we set $S = 1$ and see that the lower bound

$$\log|\Lambda^*_{A\ell}| \geq -C\cdot\log n\cdot\log\alpha\cdot\ell\cdot R_\ell$$

holds (where $R_1 = 1$ if $\ell = 1$). Thus, by comparing upper and lower bounds, we obtain

$$\min\{n_1 - n_{\ell+1}, n-m\} \leq C\ell R_\ell \log n \tag{14}$$

for $l < k$. If $\min\{n_1 - n_{\ell+1}, n-m\} = n_1 - n_{\ell+1}$, we set $R_{\ell+1} := C\ell R_\ell\log n$, otherwise we set $S_k := C\ell R_\ell \log n$. For $\ell = k$ we obtain

$$\min\{n_1, n-m\} \leq CkR_k\log n,$$

and if $\min\{n_1, n-m\} = n_1$, we set $T_{k+1} = CkT_k\log n$; otherwise we set $S_k := CkT_k\log n$.

6. Walking the steps

Depending on how large $n-m$ is compared to $n_1 - n_2, \dots, n_1 - n_k, n_1$, we do a specific series of steps, starting with Step A1.

Say we are in Step Aℓ. If $\min\{n_1 - n_{\ell+1}, n-m\} = n_1 - n_{\ell+1}$, then we get a bound $R_{\ell+1}$ for $n_1 - n_{\ell+1}$ and we continue with Step A$\ell+1$. In the other case that $n-m$ is the minimum, we get a bound $n-m \leq S_\ell$ and we continue with Step Bℓ, and after that, we continue with Steps B$\ell+1$ all the way until Step Bk. This is illustrated in Figure 2. Note that the question marks stand for numbers and letters that depend on the case, i.e. which path we are coming from. For example, if $n-m \leq n_1 - n_2$, then we do Step A1 – Step B1 – Step B2 – $\cdots$. In this case, in Step B2 we get $n_1 - n_3 \leq T_3 = C\cdot 2\cdot S_1\cdot T_2\cdot\log n$. On the other hand, if $n_1 - n_2 \leq n-m \leq n_1 - n_3$, then we do Step A1 – Step A2 – Step B2 – $\cdots$ and we get the bound $n_1 - n_3 \leq T_3 = C\cdot 2\cdot S_2\cdot R_2\cdot\log n$.

If $n_1 < n-m$, we finish with Step Ak, obtaining the bound $n_1 \leq R_{k+1}$. In all other cases, we finish with Step Bk, obtaining a bound $n_1 \leq T_{k+1}$.

In order to compute the maximal such bound $n_1 \leq R_{k+1}$ or $n_1 \leq T_{k+1}$, we need to see what exactly happens when we "walk the steps".

Case 1: $n_1 < n-m$ ("Walking down the left side").
In this case we start with $R_1 = 1$ and at each Step Aℓ we compute the next bound as described in (14), namely by

$$R_{\ell+1} = R_\ell\cdot C\cdot\ell\cdot\log n.$$

From this recursion, we immediately see that the last bound is

$$R_{k+1} = C^k\cdot k!\cdot(\log n)^k. \tag{15}$$

Case 2: $n_1 \geq n-m$ ("Crossing over").
In this case, we cross from left to right at some point in Figure 2, i.e. we go from Step Aℓ_0 to Step Bℓ_0 for some $\ell_0 \in \{1,\dots,k\}$. This means that we get bounds in the following order: $R_1 = 1, R_2, \dots, R_{\ell_0}, S_{\ell_0}, T_{\ell_0+1}, \dots, T_{k+1}$. By the same reasoning as in Case 1, the last bound on the left will be

$$R_{\ell_0} = C^{\ell_0-1}\cdot(\ell_0-1)!\cdot(\log n)^{\ell_0-1}.$$

Then, in Step Aℓ_0, the bound S_{ℓ_0} is computed in the same way, and we obtain

$$S_{\ell_0} = C^{\ell_0}\cdot\ell_0!\cdot(\log n)^{\ell_0}.$$

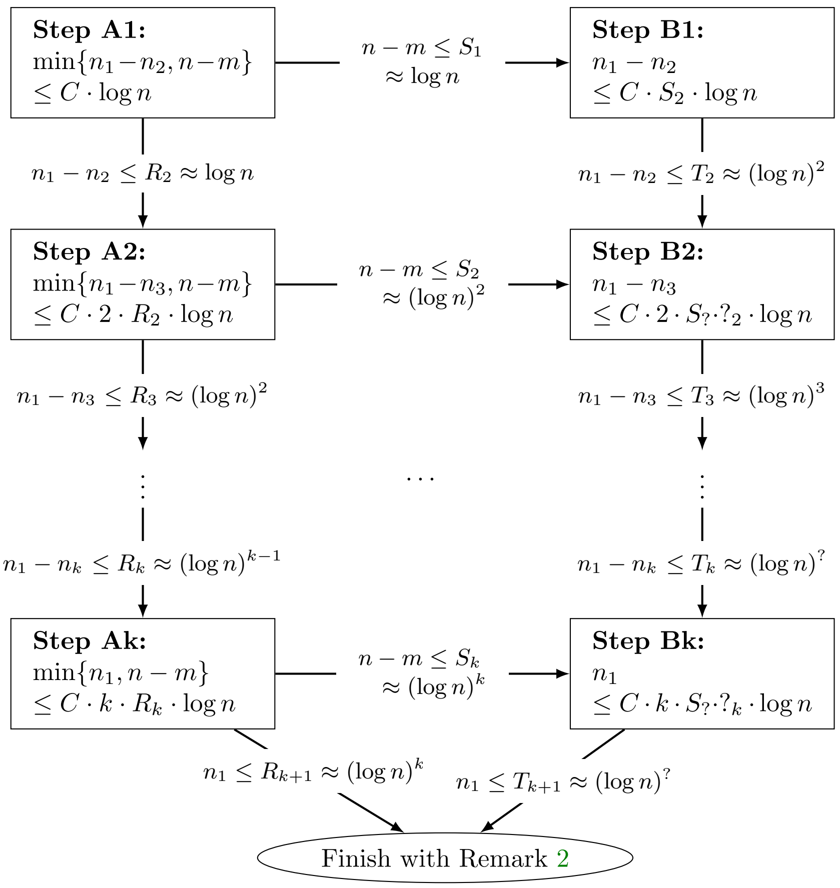

FIGURE 2. Overview of steps

After that, in Step Bℓ_0, we obtain by (13) (or in the case $\ell_0 = k$ by (12)) the bound

$$\begin{aligned} T_{\ell_0+1} &= C \cdot \ell_0 \cdot S_{\ell_0} \cdot R_{\ell_0} \cdot \log n \\ &= C \cdot \ell_0 \cdot (C^{\ell_0} \cdot \ell_0! \cdot (\log n)^{\ell_0}) \cdot (C^{\ell_0-1} \cdot (\ell_0-1)! \cdot (\log n)^{\ell_0-1}) \cdot \log n \\ &= C^{2\ell_0} \cdot (\ell_0!)^2 \cdot (\log n)^{2\ell_0}. \end{aligned}$$

After that, for $1 \leq i \leq k - \ell_0$, we have the recursion

$$\begin{aligned} T_{\ell_0+i+1} &= C \cdot (\ell_0 + i) \cdot S_{\ell_0} \cdot T_{\ell_0+i} \cdot \log n \\ &= C \cdot (\ell_0 + i) \cdot (C^{\ell_0} \cdot \ell_0! \cdot (\log n)^{\ell_0}) \cdot T_{\ell_0+i} \cdot \log n \\ &= T_{\ell_0+i} \cdot C^{\ell_0+1} \cdot \ell_0! \cdot (\log n)^{\ell_0+1} \cdot (\ell_0 + i). \end{aligned}$$

From this recursion, we obtain the last bound

$$\begin{aligned} T_{k+1} &= T_{\ell_0+1+(k-\ell_0)} \\ &= T_{\ell_0+1} \cdot (C^{\ell_0+1} \cdot \ell_0! \cdot (\log n)^{\ell_0+1})^{k-\ell_0} \cdot (\ell_0+1)(\ell_0+2)\cdots k \\ &= \left(C^{2\ell_0} \cdot (\ell_0!)^2 \cdot (\log n)^{2\ell_0}\right) \cdot C^{(\ell_0+1)(k-\ell_0)} \cdot (\ell_0!)^{k-\ell_0} \cdot (\log n)^{(\ell_0+1)(k-\ell_0)} \cdot k!/\ell_0! \\ &= k! \cdot C^{(\ell_0+1)(k-\ell_0)+2\ell_0} \cdot (\ell_0!)^{k-\ell_0+1} \cdot (\log n)^{(\ell_0+1)(k-\ell_0)+2\ell_0}. \end{aligned} \tag{16}$$

This bound depends on ℓ_0. In order to obtain an overall upper bound for all $1 \leq \ell_0 \leq k$, we first compute the maximum of the exponent $(\ell_0+1)(k-\ell_0)+2\ell_0 = k + \ell_0(k+1-\ell_0)$. For fixed k, the quadratic function $f(\ell) = k + \ell(k+1-\ell)$ has a maximum in $\ell = (k+1)/2$, and the maximum is $(k^2+6k+1)/4$. Thus we have

$$(\ell_0+1)(k-\ell_0)+2\ell_0 \leq (k^2+6k+1)/4.$$

Finding the maximum of the expression $(\ell_0!)^{k-\ell_0+1}$ is much harder. We bound the expression in a rough way. Note that for $1 \leq \ell \leq k$, we have

$$(\ell!)^{k-\ell+1} \leq \left(\ell^\ell\right)^{k-\ell+1} \leq \left(k^\ell\right)^{k-\ell+1} \leq k^{\ell(k-\ell+1)}.$$

Again, the exponent $\ell(k-\ell+1)=\ell(k+1-\ell)$ is maximal in $\ell=(k+1)/2$, and the maximum is $(k^2+2k+1)/4$. Thus we have

$$(\ell_0!)^{k-\ell_0+1} \leq k^{(k^2+2k+1)/4}.$$

Finally, for simplicity, we estimate $k! \leq k^k$. Then we obtain from (16) that

$$\begin{aligned} T_{k+1} &\leq k^k \cdot C^{(k^2+6k+1)/4} \cdot k^{(k^2+2k+1)/4} \cdot (\log n)^{(k^2+6k+1)/4} \\ &= C^{(k^2+6k+1)/4} \cdot k^{(k^2+2k+5)/4} \cdot (\log n)^{(k^2+6k+1)/4}. \end{aligned}$$

Thus, we have proven that no matter at which point we cross from the left to the right (Step Aℓ_0 – Step Bℓ_0), we always end up with the bound above. Since this bound is of course larger than the bound (15) from Case 1, we overall obtain

$$n_1 \leq C^{(k^2+6k+1)/4} \cdot k^{(k^2+2k+5)/4} \cdot (\log n)^{(k^2+6k+1)/4}. \tag{17}$$

7. Finishing the proof

Now we finish the proof as announced in Remark 2. Inequilality (17) combined with Theorem B and Lemma 1 yields

$$\begin{aligned} n &< 6 \cdot 10^{29} \cdot n_1^4 \leq 6 \cdot 10^{29} \cdot (C^{(k^2+6k+1)/4} \cdot k^{(k^2+2k+5)/4} \cdot (\log n)^{(k^2+6k+1)/4})^4 \\ &\leq C^{k^2+6k+3} \cdot k^{k^2+2k+5} \cdot (\log n)^{k^2+6k+1}, \end{aligned} \tag{18}$$

where we used $6 \cdot 10^{29} \leq C^2 = (2.1 \cdot 10^{15})^2$.

Let $\varepsilon > 0$ be given.

We want to apply Lemma 6 to inequality (18), setting $c = C^{k^2+6k+3} \cdot k^{k^2+2k+5}$ and $x = k^2+6k+1$. Moreover, we fix a $0 < \delta < 1$, which we will specify in a moment.

First, we compute the last bound from Lemma 6:

$$\begin{aligned} (2x)^{(1+\delta)x} \cdot c &= (2(k^2+6k+1))^{(1+\delta)(k^2+6k+1)} \cdot (C^{k^2+6k+3} \cdot k^{k^2+2k+5}) \\ &\leq (16k^2)^{(1+\delta)(k^2+6k+1)} \cdot C^{10k^2} \cdot k^{k^2+2k+5} \\ &\leq C_1^{k^2} \cdot k^{2(1+\delta)(k^2+6k+1)+(k^2+2k+5)}, \end{aligned}$$

where we may have set $C_1 = 16^{2\cdot 8} \cdot C^{10}$. Now if we fix a $0 < \delta < \min\{\varepsilon/2, 1\}$, then the expression $k^{(3+\varepsilon)k^2}$ grows faster than the bound $C_1^{k^2} \cdot k^{2(1+\delta)(k^2+6k+1)+(k^2+2k+5)}$. Therefore, there exists an effectively computable constant $C_2(\delta,\varepsilon)$, such that

$$(2x)^{(1+\delta)x} \cdot c \leq C_2(\delta,\varepsilon) k^{(3+\varepsilon)k^2}. \tag{19}$$

Next, we compute the second bound from Lemma 6:

$$\begin{aligned} 2^x \cdot c \cdot (\log c)^x &= 2^{k^2+6k+1} \cdot (C^{k^2+6k+3} \cdot k^{k^2+6k+1}) \cdot (\log(C^{k^2+6k+3} \cdot k^{k^2+2k+5}))^{k^2+6k+1} \\ &\leq C_3^{k^2} \cdot k^{k^2+6k+1} \cdot ((k^2+6k+1)\log C + (k^2+2k+5)\log k)^{k^2+6k+1} \\ &\leq C_4^{k^2} \cdot k^{k^2+6k+1} \cdot (k^2 \log k)^{k^2+6k+1} \\ &= C_4^{k^2} \cdot (k^3 \log k)^{k^2+6k+1}, \end{aligned}$$

where C_3, C_4 are effectively computable constants (similarly to how we obtained C_1 in the previous computation). Again, since for any fixed $\varepsilon > 0$, the expression $k^{(3+\varepsilon)k^2}$ grows faster than the bound $C_4^{k^2} \cdot (k^3 \log k)^{k^2+6k+1}$, there exists an effectively computable constant $C_5(\varepsilon)$, such that

$$2^x \cdot c \cdot (\log c)^x \leq C_5(\varepsilon) k^{(3+\varepsilon)k^2}. \tag{20}$$

Finally, we consider the first bound from Lemma 6. Since we have fixed δ, it is clear that there exists an effectively computable constant $C_6(\delta, \varepsilon)$, such that

$$\exp(\exp((1+\delta^{-1})^2)) \leq C_6(\delta, \varepsilon) k^{(3+\varepsilon)k^2}. \tag{21}$$

We set

$$C(\varepsilon) = \max\{C_2(\delta, \varepsilon), C_5(\varepsilon), C_6(\varepsilon, \delta)\}.$$

Now an application of Lemma 6 to (18), together with (21), (20) and (19), yields

$$n \leq C(\varepsilon) k^{(3+\varepsilon)k^2}. \tag{22}$$

Finally, we can bound y^a by

$$\begin{aligned} \log y^a = \log(F_n + F_m) < \log(2F_n) < \log(2\alpha^n) = \log 2 + n \log \alpha \\ \leq \log 2 + C(\varepsilon) k^{(3+\varepsilon)k^2} \log \alpha. \end{aligned}$$

This implies

$$\log y^a \leq C(\varepsilon) k^{(3+\varepsilon)k^2}$$

and we have proven Theorem 1. □

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I. Vukusic, University of Salzburg, Hellbrunnerstrasse 34/I, 5020 Salzburg, Austria
Email address: `ingrid.vukusic@plus.ac.at`

V. Ziegler, University of Salzburg, Hellbrunnerstrasse 34/I, 5020 Salzburg, Austria
Email address: `volker.ziegler@plus.ac.at`

ON A SIMPLE QUARTIC FAMILY OF THUE EQUATIONS OVER IMAGINARY QUADRATIC NUMBER FIELDS

BENJAMIN EARP-LYNCH, BERNADETTE FAYE, EVA G. GOEDHART, INGRID VUKUSIC, AND DANIEL P. WISNIEWSKI

Abstract. Let t be any imaginary quadratic integer with $|t| \geq 100$. We prove that the inequality

$$|F_t(X,Y)| = |X^4 - tX^3Y - 6X^2Y^2 + tXY^3 + Y^4| \leq 1$$

has only trivial solutions (x,y) in integers of the same imaginary quadratic number field as t. Moreover, we prove results on the inequalities $|F_t(X,Y)| \leq C|t|$ and $|F_t(X,Y)| \leq |t|^{2-\varepsilon}$. These results follow from an approximation result that is based on the hypergeometric method. The proofs in this paper require a fair amount of computations, for which the code (in Sage) is provided.

1. Introduction

In 1909, Thue [18] proved that if $F(X,Y) \in \mathbb{Z}[x,y]$ is an irreducible form of degree at least 3, and m is a nonzero integer, then the Diophantine equation (called a *Thue equation*)

$$F(x,y) = m$$

has only finitely many solutions over the integers. With the development of the theory of lower bounds for linear forms in logarithms and reduction methods, it became possible to solve specific Thue equations completely. Since the late 1980s, there exist algorithms to solve single Thue equations completely (see in particular the method of Tzanakis and De Weger [20]).

In 1990, Thomas [17] considered the parametrized family of Thue equations

$$F_t^{(3)}(X,Y) := X^3 - (t-1)X^2Y - (t+2)XY^2 - Y^3 = 1 \tag{1}$$

over integers. The resolution of (1) was completed shortly afterwards by Mignotte [15].

Since then, various families of Thue equations have been solved (see [10] for a survey from 2005).

In particular, we want to mention that the form in equation (1) is called a "simplest" cubic form, as defined by Lettl et al. [14]. In their paper, Lettl et al. moreover considered the higher degree "simplest forms"

$$F_t^{(4)}(X,Y) := X^4 - tX^3Y - 6X^2Y^2 + tXY^3 + Y^4,$$
$$F_t^{(6)}(X,Y) := X^6 - 2tX^5Y - (5t+15)X^4X^2 - 20X^3Y^3 + 5tX^2Y^4 + (2t+6)XY^5 + Y^6.$$

They solve inequalities of the shape

$$|F_t^{(i)}(X,Y)| \leq k(t),$$

where $k\colon \mathbb{Z} \to \mathbb{N}$ is a function in t, for example a linear one. Note that the fields associated with the corresponding univariate polynomials $f_t^{(i)}(X) := F_t^{(i)}(X,1)$ for $i = 3,4,6$ were already traditionally called "simplest fields." Lettl et al. [14] introduced a formal definition for simplest forms. However, their definition in fact includes more forms than they considered. The full set of simplest forms can be obtained by introducing an extra parameter in

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$F_t^{(3)}(X,Y)$, $F_t^{(4)}(X,Y)$ and $F_t^{(6)}(X,Y)$. This was pointed out by Wakabayashi [22], who then also considered those Thue inequalities.

In general, most families of Thue equations and inequalities have been considered in an integer setting, i.e. where the parameter(s) and solutions are all integers. However, there also exist results in number field settings (relative Thue equations) and in the function field setting. In the function field setting, many different families have been studied; let us just mention that simplest families have been studied in [5] and [4]. We now focus on relative Thue equations, i.e. Thue equations, where the coefficients and solutions are in a fixed number field. The name "relative" comes from the fact that if viewed as a norm form equation, the relative norm is taken.

As for classical (absolute) Thue equations, there exist algorithms for solving single relative Thue equations (in particular see [8]). However, only a few families of relative Thue equations have been studied. To the authors' best knowledge, only the following results exist so far: Families of relative Thue equations and inequalities related to $F_t^{(3)}(X,Y)$ have been solved in [11], [9] and [13]; some "non-simple" families of degree 3 and 4 have been studied in [23], [24] and [12]; and recently, Gaál et al. [7] considered the inequalities $|F_t^{(4)}(X,Y)| \leq 1$ and $|F_t^{(6)}(X,Y)| \leq 1$. All these relative Thue equations and inequalities have been studied over imaginary quadratic number fields, because those are the only number fields where integers stay away from each other and methods from the classical setting can be adapted. In particular, Heuberger [9] completely solved the family of Thue equations

$$F_t^{(3)}(X,Y) = \mu,$$

where the parameter t, the root of unity μ and the solutions x and y are integers in the same imaginary quadratic number field. Gaál et al. [7] completely solved the families $|F_t^{(4)}(X,Y)| \leq 1$ and $|F_t^{(6)}(X,Y)| \leq 1$, but only for rational integer parameters t. Their degree 4 result in particular implies the following (we exclude small integer values of t here for simplicity, as some of them lead to sporadic solutions):

Theorem A. *Let $t \in \mathbb{Z}$ with $|t| \geq 5$ and let d be a positive square-free integer. Then the inequality*

$$|F_t^{(4)}(X,Y)| \leq 1 \quad \text{in } X, Y \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})} \tag{2}$$

has only trivial solutions, i.e. solutions of the shape $(0,0)$, $(\xi,0)$ or $(0,\xi)$, where ξ is a root of unity in $\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$.

The proof of Theorem A in [7] is based on a previous paper by the same authors [6]. There they give a method for reducing the resolution of a relative Thue inequality to the resolution of the corresponding absolute Thue inequality. With this method, they are able to prove Theorem A rather quickly. However, the method only works for forms with integers coefficients.

In this paper, we want to extend Theorem A and allow the parameter t to be an imaginary quadratic integer as well. As in [14] and [9], our proof relies on the hypergeometric method. We will solve inequality (2) for imaginary quadratic integers t with $|t| \geq 100$ (see Theorem 1).

Moreover, we will prove results on some inequalities with larger upper bound than 1 (in the style of [14]), see Corollaries 1 and 2. All these results will be based on the approximation result (Proposition 1) obtained from the hypergeometric method.

The results are presented in the next section, as well as an outline of the rest of the paper (see Table 1).

Finally, let us point out that our goal is to present the proofs in full detail and also provide the used Sage code. The code is linked at the appropriate places throughout the paper and the URLs can also be found in the Appendix.

2. Results and outline of the paper

Let us set

$$F_t(X,Y) := F_t^{(4)}(X,Y) = X^4 - tX^3Y - 6X^2Y^2 + tXY^3 + Y^4$$

and let d be a positive square-free integer. We want to investigate the inequality

$$|F_t(X,Y)| \leq 1, \quad \text{in } X,Y \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}, \tag{3}$$

where $t \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ and $\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ denotes the ring of integers of the number field $\mathbb{Q}(\sqrt{-d})$.

Before we state our main result, let us split inequality (3) into equations and discuss some obvious solutions.

First, note that the absolute value of an imaginary quadratic integer is either 0 (if the integer is 0), 1 (if it is a root of unity) or larger than 1 (in all other cases). Therefore, solving (3) is equivalent to solving the two equations $F_t(X,Y) = 0$ and $F_t(X,Y) = \mu$, where μ is a root of unity in $\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$. The first equation will be solved by arguing that $F_t(X,1)$ is irreducible. The second equation will need more attention.

Next, note that

$$F_t(X,Y) = F_t(-X,-Y) = F_t(-Y,X) = F_t(Y,-X).$$

Therefore, with every solution (x,y) of (3) there usually come three more solutions $(-x,-y)$, $(-y,x)$, $(y,-x)$ and we say that these solutions are *equivalent*.

Finally, note that if ξ is a root of unity in $\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$, then $(\xi,0)$ is a solution to $F_t(X,Y) = \mu$ for $\mu = \xi^4$ and arbitrary t. We call such a solution, as well as all equivalent solutions, and the solution $(0,0)$ *trivial solutions*.

Remark 1. The only roots of unity in imaginary quadratic fields are ± 1, $\pm i$ and $\pm\zeta_6$, $\pm\zeta_6^2$, where $\zeta_6 = (1+i\sqrt{3})/2$ is the primitive sixth root of unity. Therefore, if $d \notin \{1,3\}$, we have no trivial solutions for $\mu = -1$ and we have, up to equivalence, one trivial solution for $\mu = 1$ and arbitrary t, namely $(1,0)$. If $d = 1$, we have no trivial solutions for $\mu \in \{-1, \pm i\}$ and we have, up to equivalence, two trivial solutions for $\mu = 1$, namely $(1,0)$ and $(i,0)$. If $d = 3$, we have no trivial solutions for $\mu \in \{-1, \zeta_6, -\zeta_6^2\}$, we have one trivial solutions for $\mu = 1$, one for $\mu = -\zeta_6$ and we have one trivial solution for $\mu = \zeta_6^2$, all up to equivalence and for arbitrary t. Those last three solutions are $(1,0)$, $(\zeta_6,0)$ and $(\zeta_6^2,0)$.

We will prove the following main result.

Theorem 1. *Let d be a positive square-free integer and $t \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ with $|t| \geq 100$. Then any solution $(x,y) \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ to the inequality*

$$|F_t(X,Y)| \leq 1. \tag{4}$$

is trivial, i.e. of the shape $(0,0)$ or $(\xi,0)$ or $(0,\xi)$, where ξ is a root of unity.

The typical strategy for solving Thue equations of the shape $F(X,Y) = c$ (whether with Baker's method or the hypergeometric method) uses the following fact: If (x,y) is a solution to the Thue equation, then x/y is a particularly good approximation to one of the roots of the corresponding univariate polynomial $f(X) = F(X,1)$. We say that a solution is of type j, if it approximates the j-th root.

In our case, let us set

$$f_t(X) := f_t(X,1) = X^4 - tX^3 - 6X^2 + tX + 1$$

and let α be a root of f_t. Then one can prove (in fact, this is how the simplest quartic forms were constructed in the first place) that the full set of roots of f_t is given by

$$\alpha^{(0)} = \alpha, \quad \alpha^{(1)} = \frac{\alpha-1}{\alpha+1}, \quad \alpha^{(2)} = -\frac{1}{\alpha}, \quad \alpha^{(3)} = -\frac{\alpha+1}{\alpha-1}.$$

This is further elaborated in the proof of Lemma 1. Moreover, as we will see in Lemma 4, one of the roots is close to zero and we will set $\alpha^{(0)}$ to be that root. Proposition 1 gives us a strong bound for how good general approximations to $\alpha^{(0)}$ and $\alpha^{(3)}$ can get. The proof of

Theorem 1 will mostly rely on the proposition. Note that we only consider approximations to two of the four roots. This is because we will prove in Lemma 6 that solutions of the other two types are equivalent to solutions of type 0 or 3.

Proposition 1. *Let $t \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ with $|t| \geq 100$. Let $\alpha^{(0)} = \alpha$ and $\alpha^{(3)}$ be two roots of $f_t(X)$ such that α is the unique root with $|\alpha| \leq 1/4$ and $\alpha^{(3)} = -\frac{\alpha+1}{\alpha-1}$. Then for any $p, q \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ with $|q| \geq 0.28|t|$ we have*

$$\left|\alpha^{(j)} - \frac{p}{q}\right| > \frac{1}{15.48|t||q|^{\kappa+1}}, \quad \textit{with} \quad \kappa = \frac{\log|t| + 1.08}{\log|t| - 2.59}$$

for $j \in \{0, 3\}$. In particular, since $|t| \geq 100$, we have

$$\kappa < 2.83.$$

The main strategy for proving Theorem 1 will be the following: We compare the lower bound from Proposition 1 (which is of the shape $c/(|t| \cdot |y|^{\kappa+1})$) with an upper bound, which will be of the shape $c/(|t| \cdot |y|^4)$. In other words, we will end up with an inequality of the shape $1/(|t| \cdot |y|^{\kappa+1}) \ll 1/(|t| \cdot |y|^4)$. This inequality implies an absolute upper bound on $|y|$ as soon as $\kappa < 3$.

Now note that κ can actually get arbitrarily close to 1, which means that we are "wasting" powers of $|y|$ in our proof. The next two Corollaries better show the actual power of Proposition 1. We call them Corollaries, because in contrast to Theorem 1 they follow relatively quickly from Proposition 1. This is because we exclude small solutions in the statements, whereas for Theorem 1 we have to prove lower bounds for $|y|$. The authors don't know how one could prove such general lower bounds in the context of the Corollaries.

Corollary 1. *Let $C > 0$ be given. Then there exist effectively computable constants $C_0 > 0$ and $t_0 \geq 100$, both depending on C, such that the following statement holds: For any square-free integer d and any $t \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ with $|t| \geq t_0$ the inequality*

$$|F_t(X, Y)| \leq C|t| \quad \textit{in } X, Y \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$$

has no solutions (x, y) with $\min\{|x|, |y|\} \geq C_0$, except solutions of the shape $(x, \pm x)$ with $|x| \leq (C|t|/4)^{1/4}$.

Corollary 2. *Let $0 < \varepsilon < 1$ be given. Then there exists an effectively computable constant $t_0 \geq 100$ depending on ε, such that the following statement holds: For any square-free integer d and any $t \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ with $|t| \geq t_0$ the inequality*

$$|F_t(X, Y)| \leq |t|^{2-\varepsilon} \quad \textit{in } X, Y \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$$

has no solutions (x, y) with $\min\{|x|, |y|\} > (|t|^{2-\varepsilon}/4)^{1/4}$.

In Table 1 we give an overview of the rest of the paper. Note that we focus on proving Theorem 1 throughout the paper. Only in the very last section, we generalize some of the previous results and prove the two Corollaries.

Finally, let us note that one could in principle solve inequality (4) completely, i.e. also for parameters $|t| < 100$, using Baker's method. However, in the quartic case it is not completely obvious how to find good general independent units, so solving the large number of equations requires some extra attention. Since the present paper is already rather long, the resolution for $|t| \leq 100$ is planned for future work. Some solutions for small t are mentioned in Remark 2.

3. Irreducibility and solution of $F_t(x, y) = 0$

First, we determine for which parameters t the polynomial f_t is reducible over $\mathbb{Q}(\sqrt{-d})$. Note that for solving $F_t(X, Y) = 0$, we only need to know whether $f_t(X)$ has a root in $\mathbb{Q}(\sqrt{-d})$, but reducibility may be of independent interest.

TABLE 1. Overview of paper

Section	goal
3	Determine all t's for which $f_t(X)$ is reducible; solve $F_t(X,Y)=0$.
4	Find all solutions to $F_t(X,Y)=\mu$ with $\min\{\lvert x\rvert,\lvert y\rvert\}<3$.
5	Find approximations to the roots of $f_t(X)$, e.g. $\alpha=-1/t+L(5.01\lvert t\rvert^{-3})$.
6	Establish the upper bound $\lvert x-\alpha^{(j)}y\rvert<c\lvert t\rvert^{-1}\lvert y\rvert^{-3}$ for a solution of type j; a solution (x,y) is of type j if and only if $(-y,x)$ is of type $j+2 \pmod 4$.
7	Establish lower bounds of the shape $\lvert t\rvert^k/c<\lvert y\rvert$ using Padé approximations.
8	Provide known tools and describe hypergeometric method.
9	Prove Proposition 1 with the hypergeometric method.
10	Prove Theorem 1 by combining Proposition 1 and the lower bounds from Sections 6 and 7.
11	Prove Corollaries 1 and 2 by generalizing results from Sections 6 and 7 and combining them with Proposition 1.

Lemma 1. *Let d be a positive square-free integer and $t\in\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$. Then the polynomial*

$$f_t(X)=X^4-tX^3-6X^2+tX+1$$

is reducible over $\mathbb{Q}(\sqrt{-d})$ if and only if

$$t\in\pm\{0,3,3i\pm1,4i,5i,3\sqrt{-2},2\sqrt{-3},\frac{5\sqrt{-3}\pm3}{2},\sqrt{-7},\frac{3\sqrt{-7}\pm1}{2},\sqrt{-15}\}.$$

Moreover, the polynomial $f_t(X)$ has a root in $\mathbb{Q}(\sqrt{-d})$ if and only if $t=\pm4i$.

Proof. First, note that for $t=\pm4i$ we have $f_t(X)=(X\mp i)^4$. Let us from now on assume $t\neq\pm4i$.

Next, we determine the shape of the roots of f_t. Let ϕ be the rational map $\phi\colon z\mapsto(1-z)/(1+z)$. We have $\phi^4=\mathrm{id}$ and one can check by a straight forward computation that if α is a root of f_t, then also $\phi(\alpha)$ is a root of f_t. Thus we have the roots

$$\alpha,\quad \phi(\alpha)=\frac{\alpha-1}{\alpha+1},\quad \phi^2(\alpha)=-\frac{1}{\alpha},\quad \phi^3(\alpha)=-\frac{\alpha+1}{\alpha-1}. \tag{5}$$

Now we check whether these roots are pairwise distinct. Since ϕ is cyclic of order 4, it suffices to check if $\alpha\neq\phi^2(\alpha)$. Assume that $\alpha=\phi^2(\alpha)=-1/\alpha$. Then $\alpha=\pm i$. Plugging into $f_t(X)$, we obtain $0=f_t(\pm i)=8\pm2ti$, which implies $t=\pm4i$, which we excluded. Thus, we may from now on assume that $\alpha\neq\pm i$ and that the four roots of f_t can be written as in (5).

Next, we check that f_t has no roots in $\mathbb{Q}(\sqrt{-d})$. Assume that $\alpha\in\mathbb{Q}(\sqrt{-d})$. Then since ϕ is a rational map, all roots lie in $\mathbb{Q}(\sqrt{-d})$. Moreover, being roots of the monic polynomial f_t, they are all algebraic integers. Since $\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ is integrally closed, all roots in fact lie in $\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$. But $\alpha\in\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ and $-1/\alpha\in\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ implies that α is a unit in $\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$. Therefore, we only need to check all possible units. Units in imaginary quadratic integer rings are always roots of unity, so we only need to check if $\alpha\in\{\pm1,\pm i,\pm\zeta_6,\pm\zeta_6^2\}$ can be a root of f_t. The values ±1 are not possible because $f_t(\pm1)=-4\neq0$. The case $\alpha=\pm i$ has already been handled above. For $\alpha=\pm\zeta_6$, we plug into f_t and solving $f_t(\zeta_6)=0$ for t, we obtain that $t=\pm7\sqrt{-3}/3$, which is not an imaginary quadratic integer. Also $f_t(\pm\zeta_6^2)=0$ implies that $t=\pm7\sqrt{-3}/3$, which we are not interested in. Thus, f_t has a root in $\mathbb{Q}(\sqrt{-d})$ if and only if $t=\pm4i$.

Finally, we need to check for which t the polynomial f_t factors into two irreducible polynomial over $\mathbb{Q}(\sqrt{-d})$. Assume that

$$f_t(X)=(X^2+aX+b)(X^2+cX+d),$$

with $a, b, c, d \in \mathbb{Q}(\sqrt{-d})$ and that the two factors are irreducible. Assume that α is a root of the first polynomial. Then because of the structure of the roots, we may assume without loss of generality that the second root of $X^2 + aX + b$ is either $\phi(\alpha)$ or $\phi^2(\alpha)$. We consider these two cases separately.
Case 1: The roots of $X^2 + aX + b$ are α and $\phi(\alpha) = (\alpha - 1)/(\alpha + 1)$. Then we have $b = \alpha \cdot (\alpha - 1)/(\alpha + 1)$, which implies $\alpha^2 - (b+1)\alpha - b = 0$, i.e. α is a root of $X^2 - (b+1)X - b$ with $b \in \mathbb{Q}(\sqrt{-d})$. This contradicts the fact that the unique minimal polynomial of α is $X^2 + aX + b$.
Case 2: The roots of $X^2 + aX + b$ are α and $\phi^2(\alpha) = -1/\alpha$. Then we have $b = \alpha \cdot (-1/\alpha) = -1$ and consequently also $d = -1$. Thus we get

$$\begin{aligned} X^4 - tX^3 - 6X^2 + tX + 1 &= (X^2 + aX - 1)(X^2 + cX - 1) \\ &= X^4 + (a+c)X^3 + (ac-2)X^2 - (a+c)X + 1 \end{aligned}$$

and comparing coefficients we obtain $a + c = -t$ and $ac = -4$. By Vietá's formula, these two equations imply that a and c are the roots of the polynomial $X^2 + tX - 4 = 0$. This implies that a and c are integral over $\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ and therefore in $\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$. Moreover, since $ac = -4$, we have that $|a|^2$ and $|c|^2$ both divide 16. In particular, $|a|$ is bounded by 4 and we can list all such imaginary quadratic integers (see the Appendix for how to find them in a systematic way). Then for each $a \neq 0$ we check whether $|a|^2$ divides 16 and if so, we compute $c = -4/a$. Then, if c is integral , we compute $t = -(a + c)$. Running these computations in Sage [16] takes less than a second and yields exactly the list of exceptional t's that is stated in the lemma. Note that if we consider $-a$ instead of a, we end up with $-c$ and $-t$, so for the computations it suffices to consider only a's that are either positive or have positive imaginary part. The Sage code for this proof can be found here. □

Now we immediately get the following result:

Lemma 2. *Let d be a positive square-free integer, $t \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ and $t \neq \pm 4i$. Then the equation*

$$F_t(X, Y) = X^4 - tX^3Y - 6X^2Y^2 + tXY^3 + Y^4 = 0 \quad \text{in } X, Y \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})} \tag{6}$$

has only the trivial solution $(x, y) = (0, 0)$.

Proof. Let $(x, y) \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ be a solution to (6). If $y = 0$, then it is easy to see that $x = 0$, i.e. (x, y) is the trivial solution. Otherwise, we have $0 = F_t(x, y) = y^4 f_t(x/y)$, i.e. $x/y \in \mathbb{Q}(\sqrt{-d})$ is a root of f_t, which is impossible by Lemma 1. □

Thus, in order to solve $|F_t(X, Y)| \leq 1$, we can from now on focus on the equation $F_t(X, Y) = \mu$, where μ is a root of unity in $\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$.

4. Small solutions

For technical reasons we will want to assume $\min\{|x|, |y|\} \geq 3$ later in the proof. Therefore, we now find all solutions for all $|t| \geq 100$ with $\min\{|x|, |y|\} < 3$. The proof is based on the idea in [11, Proof of Lemma 5].

Lemma 3. *Let d be a positive square-free integer, $t \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ with $|t| \geq 100$ and $\mu \in \mathbb{Z}^\times_{\mathbb{Q}(\sqrt{-d})}$. Let $(x, y) \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ be a solution to the equation*

$$F_t(X, Y) = \mu. \tag{7}$$

If $\min\{|x|, |y|\} < 3$, then (x, y) is trivial, i.e. of the shape $(\xi, 0)$ or $(0, \xi)$, where ξ is a root of unity.

Proof. If $x = 0$ or $y = 0$, we see immediately that (x, y) is trivial by plugging into (7). Let us from now on assume that $x \neq 0$ and $y \neq 0$.

Since the solutions (x, y) and $(-y, x)$ are equivalent, we may assume without loss of generality that $|x| \leq |y|$ and in particular $0 < |x| < 3$. Moreover, since (x, y) and $(-x, -y)$

are equivalent, we may assume without loss of generality that either the imaginary part of x is $\Im(x) > 0$ or that $x \in \mathbb{Z}$ and $x > 0$. There are only finitely many such imaginary quadratic integers x with $0 < |x| < 3$ and we can list them, see the Appendix for more details. For each of these values of x, we now describe how to find all (x, y) that are a solution to (7) for some t and μ.

Assume that (x, y) is a solution to (7) with $0 < |x| < 3$. Then (7) implies that

$$y(y^3 + txy^2 - 6x^2y - tx^3) = \mu - x^4. \tag{8}$$

We consider two cases.

Case 1: $\mu - x^4 \neq 0$. Since all elements are imaginary quadratic integers, we get from (8) that

$$|y| = |y| \cdot 1 \leq |y| \cdot |y^3 + txy^2 - 6x^2y - tx^3| = |\mu - x^4| \leq |\mu| + |x|^4$$

and thus $|y| \leq 1 + |x|^4$.

Case 2: $\mu - x^4 = 0$. Since we are assuming $y \neq 0$, equation (8) implies

$$y^3 + txy^2 - 6x^2y - tx^3 = 0,$$

which is equivalent to

$$(tx + y)(y - x)(x + y) = 5x^2y. \tag{9}$$

Noting that $|x| = |\mu|^{1/4} = 1$, we obtain from (9) that

$$5|y| = 5|x|^2|y| = |tx + y||y - x||x + y| \geq 1 \cdot (|y| - |x|)(|y| - |x|) = (|y| - 1)^2,$$

which implies $|y| < 6.86$.

Thus, in both cases we have a small upper bound on $|y|$ and there are only finitely many such imaginary quadratic integers y and we can list them. Note that, if $x \notin \mathbb{Z}$, then the quadratic integers y are in the fixed imaginary quadratic field $\mathbb{Q}(x)$. Moreover, one can check that $f_t(x, -y) = f_{-t}(x, y)$ for any t, x, y, and we can therefore restrict our search to y with either $\Im(y) > 0$ or $y \in \mathbb{Z}$ and $y > 0$.

Then we only need to plug all such (x, y) into (7) and compute

$$t = \frac{x^4 - 6x^2y^2 + y^4 - \mu}{x^3y - xy^3}$$

and check if it might be a quadratic integer. Here we need to check all possible units μ that are in the numberfield of x and y. Note that if $(x, y) \in \mathbb{Z}^2$, then we check all roots of unity ± 1, $\pm i$, $\pm\zeta_6$, $\pm\zeta_6^2$.

Now if t is a quadratic integer and $|t| \geq 100$, then we have found a non-trivial solution of interest. Otherwise, (x, y) cannot be a non-trivial solution of a considered equation.

Doing all the computations with Sage [16] takes a few seconds on a usual pc and reveals no non-trivial solutions (see here). □

Remark 2. If we drop the assumption $|t| \geq 100$ in the above described computer search, we get nontrivial solutions for

$$t \in \pm\{4i, 3\sqrt{-2}, 2\sqrt{-3}\} \cup \pm\{1, 4, \frac{\sqrt{-3} \pm 1}{2}, \sqrt{-17}\}.$$

For the t's in the first set we have by Lemma 1 that f_t is reducible, while for the t's in the second set it is irreducible.

5. Approximation of the roots

Let $\alpha = \alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}$ be the roots of f_t. As described in the proof of Lemma 1, we may write

$$\alpha^{(0)} = \alpha, \quad \alpha^{(1)} = \frac{\alpha - 1}{\alpha + 1}, \quad \alpha^{(2)} = -\frac{1}{\alpha}, \quad \alpha^{(3)} = -\frac{\alpha + 1}{\alpha - 1}. \tag{10}$$

We now compute asymptotic estimates for the roots of f_t. This can be done e.g. with Sage [16], substituting $1/t =: s$ and doing the computations in the ring of power series with variable s. One can approximate α by starting at $x_0 = 0$ and applying several steps of Newton's Method. Then the other approximations can be obtained from the formulas in (10). The error terms follow from an application of Rouché's Theorem, see the proof below.

We use the following L-notation: For functions h, k we write $h(z) = L(k(z))$ if $|h(z)| \leq k(z)$ for all $z \geq 100$.

Lemma 4. *Let $t \in \mathbb{C}^\times$ with $|t| \geq 100$. Then the roots of $f_t(X) = X^4 - tX^3 - 6X^2 + tX + 1$ are approximated by*

$$\alpha^{(0)} = -\frac{1}{t} + L\left(\frac{5.01}{|t|^3}\right), \qquad \alpha^{(2)} = t + L\left(\frac{5.02}{|t|}\right),$$

$$\alpha^{(1)} = -1 + L\left(\frac{2.16}{|t|}\right), \qquad \alpha^{(3)} = 1 + L\left(\frac{2.16}{|t|}\right).$$

Proof. We want to prove the approximation for $\alpha^{(0)}$ via Rouché's Theorem. Let us set $h(z) := f(-\frac{1}{t} + z)$. The goal is to show that $h(z)$ has a root in the disc with origin 0 and radius $5.01|t|^{-3}$. We check that $|h(0)| < |h(z) - h(0)|$ for any z with $|z| = 5.01|t|^{-3}$. On the one hand we have

$$|h(0)| = \left|f(-\frac{1}{t})\right| = \left|-\frac{5}{t^2} + \frac{1}{t^4}\right| \leq \frac{5.0001}{|t|^2},$$

where we used $|t| \geq 100$ for the estimate. On the other hand we have

$$\begin{aligned}
|h(z) - h(0)| &= \left|f(-\frac{1}{t} + z) - f(-\frac{1}{t})\right| \\
&= \left|z^4 + \left(-t - \frac{4}{t}\right)z^3 + \left(-3 + \frac{6}{t^2}\right)z^2 + \left(t + \frac{9}{t} - \frac{4}{t^3}\right)z\right| \\
&\geq |tz| - \left|z^4 + \left(-t - \frac{4}{t}\right)z^3 + \left(-3 + \frac{6}{t^2}\right)z^2 + \left(\frac{9}{t} - \frac{4}{t^3}\right)z\right| \\
&> |t| \cdot 5.01|t|^{-3} - 50|t|^{-4} > \frac{5.0001}{|t|^2},
\end{aligned}$$

where we used $|z| = 5.01|t|^{-3}$ and $|t| \geq 100$. Overall, we have obtained

$$|h(0)| < |h(z) - h(0)| \quad \text{for } |z| = 5.01|t|^{-3}.$$

Now note that the function $h_1(z) := h(z) - h(0)$ has a root with $|z| < 5.01|t|^{-3}$, namely $z = 0$. Moreover, $h_1(z)$ and the constant function $h(0)$ are holomorphic. Thus Rouché's Theorem tells us that the function $h_1(z) + h(0) = h(z) = f(-\frac{1}{t} + z)$ has a root with $|z| < 5.01|t|^{-3}$. This immediately implies that f has a root with value $-\frac{1}{t} + L\left(\frac{5.01}{|t|^3}\right)$.

The other approximations can be checked analogously, see the code for details. □

6. Types of solutions

As in Section 4, let d be a positive square-free integer and we continue focusing on the Thue equations of the type

$$F_t(X, Y) = \mu, \tag{11}$$

where $t \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ and $\mu \in \mathbb{Z}^\times_{\mathbb{Q}(\sqrt{-d})}$.

In general, when solving Thue equations, one of the tricks is to use the following fact: A solution (x, y) usually corresponds to an extremely good approximation to one of the roots of the related univariate polynomial.

Let $(x, y) \in \mathbb{Z}^2_{\mathbb{Q}(\sqrt{-d})}$ be a solution to equation (11). Then we define

$$\beta^{(i)} := x - \alpha^{(i)} y, \quad \text{for } i = 0, 1, 2, 3.$$

We say that (x, y) is a solution of *type* j if

$$|\beta^{(j)}| = \min_{0 \leq i \leq 3} |\beta^{(i)}|.$$

The next lemma quantifies how good such an approximation is.

Lemma 5. *Let, $t \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$, $|t| \geq 100$ and $\mu \in \mathbb{Z}^{\times}_{\mathbb{Q}(\sqrt{-d})}$. Let $(x, y) \in \mathbb{Z}^2_{\mathbb{Q}(\sqrt{-d})}$ be a non-trivial solution to the equation*

$$F_t(X, Y) = \mu.$$

If (x, y) is of type j, then

$$|x - \alpha^{(j)} y| = |\beta^{(j)}| < \frac{8.86}{|t| \cdot |y|^3}.$$

Proof. Let (x, y) be a solution of type j. From

$$|y||\alpha^{(i)} - \alpha^{(j)}| = |\alpha^{(i)} y - x + x - \alpha^{(j)} y| \leq |\beta^{(i)}| + |\beta^{(j)}| \leq 2|\beta^{(i)}|$$

and

$$\prod_{i=0}^{3} \beta^{(i)} = \prod_{i=0}^{3} (x - \alpha^{(i)} y) = F_t(x, y) = \mu,$$

we conclude that

$$|\beta^{(j)}| = \frac{1}{\prod_{i \neq j} |\beta^{(i)}|} \leq \frac{2^3}{|y|^3 \prod_{i \neq j} |\alpha^{(j)} - \alpha^{(i)}|}.$$

From Lemma 4 we see (assuming $|t| \geq 100$) that the difference between any two roots is at least 0.96.

Moreover, the difference between $\alpha^{(2)}$ and any other root is at least $0.98|t|$. Thus we obtain

$$|\beta^{(j)}| \leq \frac{1}{|y|^3} \cdot \frac{8}{0.96^2 \cdot 0.98|t|} < \frac{8.86}{|t| \cdot |y|^3}.$$

□

Finally, we describe how the types of equivalent solutions (x, y) and $(-y, x)$ relate.

Lemma 6. *Let $t \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$, $|t| \geq 100$ and $\mu \in \mathbb{Z}^{\times}_{\mathbb{Q}(\sqrt{-d})}$. Let $(x, y) \in \mathbb{Z}^2_{\mathbb{Q}(\sqrt{-d})}$ be a solution to the equation*

$$F_t(X, Y) = \mu$$

with $\min\{|x|, |y|\} \geq 3$. Then (x, y) is of type j if and only if $(-y, x)$ is a solution of type $j + 2 \pmod 4$.

Proof. Let $|t| \geq 100$. From Lemma 5 it follows that

$$\left| \alpha^{(j)} - \frac{x}{y} \right| < \frac{8.86}{|t| \cdot |y|^4} < 0.09$$

for any non-trivial solution (x, y) of any type j. Since by Lemma 4 the distance between any two distinct roots is at least 0.96, we can say that any non-trivial solution is of type j if and only if $|\alpha^{(j)} - x/y| < 0.48$.

Now let (x, y) be a non-trivial solution of type j. Then from Lemma 5 we get (with $j+2$ computed in modulo 4 arithmetic) that

$$\left| \alpha^{(j+2)} - \frac{-y}{x} \right| = \left| -\frac{1}{\alpha^{(j)}} + \frac{y}{x} \right| = \left| \frac{-x + \alpha^{(j)} y}{\alpha^{(j)} x} \right| < \frac{8.86}{|t| \cdot |y|^3 \cdot |x| \cdot |\alpha^{(j)}|}.$$

Since by Lemma 4 we have $|\alpha^{(j)}| > 0.94|t|^{-1}$, we obtain

$$\left| \alpha^{(j+2)} - \frac{-y}{x} \right| < \frac{9.43}{|x| \cdot |y|^3}.$$

With $\min\{|x|, |y|\} \geq 3$ we obtain

$$\left| \alpha^{(j+2)} - \frac{-y}{x} \right| < 0.48,$$

which implies that $(-y, x)$ is indeed of type $j + 2 \pmod 4$. □

In view of Lemma 6, since (x, y) and $(-y, x)$ are equivalent solutions, it is enough to consider only solutions of type 0 or of type 3 in order to prove Theorem 1.

7. Finding lower bounds for $|y|$

In this section we prove the following lower bounds for $|y|$. Note that we could push the absolute lower bound even higher, but the goal is only to contradict the upper bound that will follow from Proposition 1.

Lemma 7. *Let, $t \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$, $|t| \geq 100$ and $\mu \in \mathbb{Z}^{\times}_{\mathbb{Q}(\sqrt{-d})}$. Let $(x, y) \in \mathbb{Z}^2_{\mathbb{Q}(\sqrt{-d})}$ be a solution to the equation*

$$F_t(X, Y) = \mu. \tag{12}$$

Assume that $\min\{|x|, |y|\} \geq 3$ and that (x, y) is a solution of type 0 or 3. Then we have

$$|y| > 0.44|t| \tag{13}$$

and moreover

$$|y| > 1.047 \cdot 10^{13}. \tag{14}$$

In fact, we will do several steps and prove bounds of the shape

$$|y| > \frac{|t|^k}{c},$$

for larger and larger integers k. This is based on the ideas in [23, Section 5]. First, we consider the case where the solution is of type 0. Afterwards, we do analogous computations for the type 3 case.

7.1. Lower bound for y of type 0

Let $|t| \geq 100$ and $(x, y) \in \mathbb{Z}^2_{\mathbb{Q}(\sqrt{-d})}$ be a solution of type 0 and $\min\{|x|, |y|\} \geq 3$.

Step 1: We combine Lemma 5 and the approximation of $\alpha = \alpha^{(0)}$ from Lemma 4:

$$|x| - \frac{1.01}{|t|}|y| \leq \left| x - \left(-\frac{1}{t} + L\left(\frac{5.01}{|t|^3}\right)\right) y \right| = |x - \alpha y| < \frac{8.86}{|t| \cdot |y|^3}.$$

Since we are assuming $\min\{|x|, |y|\} \geq 3$, this implies

$$3 \leq |x| < \frac{1.01}{|t|}|y| + \frac{8.86}{|t| \cdot |y|^3} < \frac{1.12}{|t|}|y|$$

and we get

$$|y| > 2.67 \cdot |t|.$$

In particular, we have proven equation (13) in the type 0 case.

Step 2: As in the previous step, we combine Lemma 4 and Lemma 5, but now we multiply the expressions with an extra factor $|t|$:

$$|tx + y| - \frac{5.01}{|t|^2}|y| \leq \left| tx - \left(-1 + L\left(\frac{5.01}{|t|^2}\right)\right) y \right| = |tx - t\alpha y| < \frac{8.86}{|y|^3}.$$

If $tx + y = 0$, then plugging $y = -tx$ into equation (12) yields $x^4(1 - 5t^2) = \mu$. Considering the absolute values, this immediately leads to a contradiction for $t \neq 0$. Thus we may assume that $tx + y$ is a non-zero element of $\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$, which means that it has absolute value at least 1 and we obtain from the above inequality

$$1 \leq |tx + y| < \frac{5.01}{|t|^2}|y| + \frac{8.86}{|y|^3} < \frac{5.01}{|t|^2}|y| + \frac{8.86}{(2.67 \cdot |t|)^4}|y| < \frac{5.02}{|t|^2}|y|,$$

which implies

$$|y| > \frac{|t|^2}{5.02}.$$

The trick in Step 2 to multiply the equation with $|t|$ worked because of the gap in the exponents of $|t|$ in the main and the L-term in the approximation of α. This is now exhausted and we need a new idea. We use the idea from [23, Proof of Lemma 7], which is to replace the series expansion of α by Padé approximations. Moreover, we will need a higher precision approximation of α. Analogously to Lemma 4 one can prove (see code) the following.

Lemma 8. *Let $B(1/t)$ be the approximation to α obtained by applying 15 steps of Newton's method and with precision $O(|t|^{-31})$, i.e.*

$$B(1/t) = -t^{-1} + 5t^{-3} - 46t^{-5} + 509t^{-7} + \cdots + 1821914025180536t^{-29}.$$

Then we have

$$|\alpha - B(1/t)| < \frac{2.71 \cdot 10^{16}}{|t|^{31}}.$$

To compute Padé approximations $A(1/t) = U(1/t)/V(1/t)$ to $B(1/t)$ we can use Sage's power series method `pade()` with the variable $s := 1/t$.
Step 3: In Step 3 we compute the Padé approximation where the degrees of U and V are bounded by 2. We obtain

$$\alpha \approx B(1/t) \approx A(1/t) = \frac{U(1/t)}{V(1/t)} = \frac{-\frac{1}{t}}{\frac{5}{t^2} + 1}.$$

The constant that we will obtain in this step will mostly depend on the quality of this approximation. Therefore, we first compute the approximation error in the following sense:

$$|U(1/t) - B(1/t)V(1/t)| = \left| \frac{21}{t^5} - \frac{279}{t^7} + \dots \right| < \frac{21.028}{|t|^5},$$

where the dots stand for a finite expression, which was actually computed and then estimated using $|t| \geq 100$.

Next, we get x and y involved and compute some other approximations. First, we use Lemma 5 and the bound from Step 2:

$$|x - \alpha y| < \frac{8.86}{|t| \cdot |y|^3} < \frac{8.86}{|t|(|t|^2/5.02)^4}|y| < \frac{5627}{|t|^9}|y|.$$

With this estimate and Lemma 8 we get

$$|x - B(1/t)y| \leq |x - \alpha y| + |\alpha - B(1/t)| \cdot |y| < \frac{5627}{|t|^9}|y| + \frac{2.71 \cdot 10^{16}}{|t|^{31}}|y|.$$

We need one more intermediate estimate:

$$|V(1/t)| = \left| 1 + \frac{5}{t^2} \right| < 1.01.$$

Finally, we combine these estimates to find a useful upper bound for an imaginary quadratic integer:

$$\begin{aligned}
|(t^2+5)x + ty| &= |t^2 V(1/t)x - t^2 U(1/t)y| \\
&= \left| t^2 V(1/t)x - t^2 \left(V(1/t)B(1/t) + L\left(\frac{21.028}{|t|^5}\right)\right) y \right| \\
&\leq \frac{21.028}{|t|^3}|y| + |t|^2 |V(1/t)||x - B(1/t)y| \\
&< \frac{21.028}{|t|^3}|y| + |t|^2 \cdot 1.01 \cdot \left(\frac{5627}{|t|^9} + \frac{2.71 \cdot 10^{16}}{|t|^{31}}\right)|y| \\
&< \frac{21.028}{|t|^3}|y| + \left(\frac{5684}{|t|^7} + \frac{2.74 \cdot 10^{16}}{|t|^{29}}\right)|y| \\
&< \frac{21.03}{|t|^3}|y|.
\end{aligned} \tag{15}$$

Assume for a moment that the expression on the left vanishes. Then we have $x = -t/(t^2 + 5)y$. Computing $F_t(x, y)$ and using $|y| > |t|^2/5.02$ we obtain

$$
\begin{aligned}
(16)\qquad |F_t(x,y)| &= \left|F_t\left(\frac{-t}{t^2+5}y, y\right)\right| = \left|f_t\left(\frac{-t}{t^2+5}\right)\right| \cdot |y|^4 \\
&= \left|\frac{21t^4+225t^2+625}{(t^2+5)^4}\right| \cdot |y|^4 > \frac{21|t|^4-225|t|^2-625}{(|t|^2+5)^4}\left(\frac{|t|^2}{5.02}\right)^4.
\end{aligned}
$$

For $|t| \geq 100$ the last expression is larger than 1, which is a contradiction.

Thus we may assume that the expression bounded in (15) is non-zero. Since it is clearly an imaginary quadratic integer, it has absolute value at least 1. This yields

$$1 < \frac{21.03}{|t|^3}|y|,$$

which implies

$$|y| > \frac{|t|^3}{21.03}.$$

Steps 4–9: At each Step k $(4 \leq k \leq 9)$ we start with a bound of the form

$$|y| > \frac{|t|^{k-1}}{c_0},$$

where in Step 4 we have $c_0 = 21.03$. Then we compute a Padé approximation $\alpha \approx A(1/t) = U(1/t)/V(1/t)$ where $U, V \in \mathbb{Z}[X]$ are polynomials of degree at most $k - 1$.

Analogously to Step *3*, it we can then compute a constant c_1 such that

$$|U(1/t) - B(1/t)V(1/t)| \leq \frac{c_1}{|t|^{2k-1}}.$$

Next, we can estimate

$$|x - \alpha y| < \frac{8.86}{|t|\cdot|y|^3} < \frac{8.86}{|t|(|t|^{k-1}/c_0)^4}|y| = \frac{c_2}{|t|^{4k-3}}|y|,$$

with $c_2 = 8.86 \cdot c_0^4$, and

$$|x - B(1/t)y| < \left(\frac{c_2}{|t|^{4k-3}} + \frac{2.71\cdot 10^{16}}{|t|^{31}}\right)|y|.$$

Finally, we compute an upper bound

$$|V(1/t)| \leq c_3.$$

Since we are assuming $|t| \geq 100$, it turns out that d is always roughly the size of the constant term in V.

Then, by an analogous computation to (15) we obtain

$$(17)\qquad |t^{k-1}V(1/t)x - t^{k-1}U(1/t)y| < \left(\frac{c_1}{|t|^k} + \frac{c_2c_3}{|t|^{3k-2}} + \frac{2.71\cdot 10^{16}\cdot c_3}{|t|^{31+1-k}}\right)|y| \leq \frac{c}{|t|^k}|y|,$$

where $c = c_1 + c_2c_3 \cdot 100^{-(2k-2)} + 2.71 \cdot 10^{16} \cdot c_3 \cdot 100^{-(32-2k)}$, which is roughly of the size of c_1.

To finish the step, we only need to check that the left hand side cannot vanish. If it did, we would have $x = A(1/t)y$. To show that this is impossible, we do an analogous computation to (16):

$$
\begin{aligned}
|F_t(x,y)| = \left|f_t\left(\frac{t^{k-1}U(1/t)}{t^{k-1}V(1/t)}\right)\right| \cdot |y|^4 &= \left|\frac{P(t)}{(t^{k-1}V(1/t))^4}\right| \cdot |y|^4 \\
&> \frac{|P(t)|}{|t^{k-1}V(1/t)|^4}\left(\frac{|t|^{k-1}}{c_0}\right)^4.
\end{aligned}
$$

It turns out that $P(t) = F_t(t^{k-1}U(1/t), t^{k-1}V(1/t))$ is always a polynomial of degree $2k - 2$ and $t^{k-1}V(1/t)$ is a polynomial of degree $k - 1$. Thus, we have a lower bound of the order

$|t|^{2k-2}$. Computing the coefficients and estimating using $|t| \geq 100$, one can indeed show in every step that $|F_t(A(1/t)y, y)| > 1$, a contradiction.

Thus we may assume that the left hand side of (17) is at least 1, which implies

$$|y| > \frac{|t|^k}{c} \tag{18}$$

and we set $c_0 = c$ for the next step.

Always rounding up with a 4-digit precision when computing c, we obtain the values presented in Table 2 (see code). Moreover, the table shows the lower bounds for $|y|$ that follow immediately from setting $|t| = 100$ in (18). The lower bound for $|y|$ obtained in Step 9 proves equation (14) of Lemma 7 in the type 0 case.

Table 2. Constants obtained in Steps 4–9 for type 0

k	4	5	6	7	8	9	10	11
c	429.8	2436	4210	$1.863 \cdot 10^5$	$3.242 \cdot 10^6$	$5.915 \cdot 10^6$	$8.066 \cdot 10^7$	$4.726 \cdot 10^8$
$\lvert y\rvert > \dots$	$2.327 \cdot 10^5$	$4.105 \cdot 10^6$	$2.375 \cdot 10^8$	$5.369 \cdot 10^8$	$3.084 \cdot 10^9$	$1.691 \cdot 10^{11}$	$1.240 \cdot 10^{12}$	$2.116 \cdot 10^{13}$

7.2. Lower bound for y of type 3

Now assume that (x, y) with $\min\{|x|, |y|\} \geq 3$ is a solution of type 3. We proceed analogously to the above subsection.

Step 1: We combine Lemma 5 and the approximation of $\alpha^{(3)}$ from Lemma 4:

$$|x-y| - \frac{2.16}{|t|}|y| \leq \left| x - \left(1 + L\left(\frac{2.16}{|t|}\right)\right) y \right| = |x - \alpha^{(3)} y| < \frac{8.86}{|t| \cdot |y|^3}.$$

Since we are assuming $|y| \geq 3$, this implies

$$|x-y| \leq \frac{2.16}{|t|}|y| + \frac{8.86}{|t| \cdot |y|^3} < \frac{2.27}{|t|}|y|.$$

If $x = y$, then plugging into $F_t(x, y) = \mu$ yields $-4x^4 = \mu$, which is impossible. Thus we have $|x - y| \geq 1$ and the above inequality implies

$$|y| > \frac{|t|}{2.27} > 0.44|t|.$$

Thus we have proven equation (13) in the type 3 case.

For the further steps, we need a higher precision approximation for $\alpha^{(3)}$. We can compute it in Sage via $\alpha^{(3)} = -(\alpha+1)/(\alpha-1)$, where we use an approximation to α obtained with Newton's method. Analogously to Lemma 4 and Lemma 8, one can prove (see code) the following.

Lemma 9. *Let $B_3(1/t)$ be the approximation to $\alpha^{(3)}$ given by*

$$B_3(1/t) = 1 - 2t^{-1} + 2t^{-2} + 8t^{-3} - 18t^{-4} + \cdots + 1435829041889280t^{-29}.$$

Then we have

$$|\alpha - B_3(1/t)| < \frac{9.84 \cdot 10^{15}}{|t|^{30}}.$$

The remaining steps are analogous to Step *3* and Steps *4–9* from the previous subsection. We only need to replace α by $\alpha^{(3)}$, $B(1/t)$ by $B_3(1/t)$ and $2.71 \cdot 10^{16}/|t|^{31}$ by $9.84 \cdot 10^{15}/|t|^{30}$.

Steps 2–10: We start with $k = 2$ and $c_0 = 2.27$. The results from each step (see code) are presented in Table 3. The lower bound for $|y|$ obtained in Step 10 proves equation (14) of Lemma 7 in the type 3 case. Thus we have completed the proof of Lemma 7.

TABLE 3. Constants obtained in Steps 4–9 for type 3

k	2	3	4	5	6	7	8	9	10	11
c	10.14	42.48	868.0	4921	8503	$3.762 \cdot 10^5$	$6.549 \cdot 10^6$	$1.195 \cdot 10^7$	$1.629 \cdot 10^8$	$9.547 \cdot 10^8$
$\lvert y\rvert > \dots$	986.2	23540	$1.152 \cdot 10^5$	$2.032 \cdot 10^6$	$1.176 \cdot 10^8$	$2.658 \cdot 10^8$	$1.527 \cdot 10^9$	$8.370 \cdot 10^{10}$	$6.138 \cdot 10^{11}$	$1.047 \cdot 10^{13}$

8. The hypergeometric method

The hypergeometric method has its name from the hypergeometric function, which in this context is used to construct very good approximations of a root α of $f_t(X)$. With these approximations one can then obtain an effective irrationality measure for α by the following elementary but ingenious lemma. We have borrowed it from [9, Lemma 2.7], which is a generalization of [1, Lemma 2.8]. However, the idea is much older and goes back to Thue and Siegel, see [3] for a historic overview.

Lemma A. *Let $\alpha \in \mathbb{C}$ and suppose that there exist real numbers $k_0, l_0 > 0$ and $E, Q > 1$ such that for all positive integers r there are integers $p_r, q_r \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ with $|q_r| < k_0 Q^r$ and $|q_r\alpha - p_r| \leq l_0 E^{-r}$ satisfying $p_r q_{r+1} \neq p_{r+1} q_r$ for all r. Then for any integers $p, q \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ with $|q| \geq 1/(2l_0)$, we have*

$$\left|\alpha - \frac{p}{q}\right| > \frac{1}{c|q|^{\kappa+1}}, \quad \text{where} \quad c = 2k_0 Q (2l_0 E)^{\kappa} \quad \text{and} \quad \kappa = \frac{\log Q}{\log E}.$$

Now we discuss how to obtain such good approximations to a root α of $f_t(X)$. We will construct sequences of polynomials $\mathbf{A}_r, \mathbf{B}_r \in \mathbb{Q}(\sqrt{-d})[X]$ such that $|\alpha \mathbf{A}_r(\xi) - \mathbf{B}_r(\xi)|$ is very small if ξ is close to α. This will almost give us the approximations that we need for the application of Lemma A. We will just need to choose ξ appropriately and clear denominators with numbers M_r, so that $q_r = M_r \mathbf{A}_r(\xi)$ and $p_r = M_r \mathbf{B}_r(\xi)$ are indeed in $\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$.

Let ${}_2\mathcal{F}_1$ denote the classic *hypergeometric function*. For positive integers n, r set

$$\chi_{n,r}(X) = {}_2\mathcal{F}_1\left(-r, -r - \frac{1}{n}; 1 - \frac{1}{n}; X\right) \in \mathbb{Q}[X],$$
$$\chi^*_{n,r}(X, Y) = Y^r \chi_{n,r}\left(\frac{X}{Y}\right) \in \mathbb{Q}[X, Y].$$

Note that $\chi_{n,r}$ is a polynomial of degree r and $\chi^*_{n,r}$ is its homogenization.

The basis for constructing $\mathbf{A}_r$ and $\mathbf{B}_r$ is Thue's "Fundamentaltheorem" [19]. We use the version that is stated in [9, Lemma 2.1].

Lemma B (Thue)**.** *Let K be a field of characteristic 0, $\mathbf{P} \in K[X]$ be a square-free polynomial of degree $n \geq 2$ and assume that there is a square-free quadratic polynomial $\mathbf{U} \in K[X]$ such that*

$$\mathbf{U}\mathbf{P}'' - (n-1)\mathbf{U}'\mathbf{P}' + \frac{n(n-1)}{2}\mathbf{U}''\mathbf{P} = 0 \tag{19}$$

holds, where the prime denotes differentiation with respect to the indeterminate X. We set $\lambda = \frac{1}{4}$disc$(\mathbf{U})$, where disc$(\mathbf{U}) = \mathbf{U}'^2 - 2\mathbf{U}\mathbf{U}'' \in K$ *is the discriminant of* $\mathbf{U}$. *We define the polynomials contained in $K(\sqrt{\lambda})[X]$ by*

$$\mathbf{Y} = 2\mathbf{U}\mathbf{P}' - n\mathbf{U}'\mathbf{P},$$
$$\mathbf{a} = \frac{n^2-1}{6}(\sqrt{\lambda}\mathbf{U}' + 2\lambda), \qquad \mathbf{c} = \frac{n^2-1}{6}(\sqrt{\lambda}(\mathbf{U}'X - 2\mathbf{U}) + 2\lambda X),$$
$$\mathbf{b} = \frac{n^2-1}{6}(\sqrt{\lambda}\mathbf{U}' - 2\lambda), \qquad \mathbf{d} = \frac{n^2-1}{6}(\sqrt{\lambda}(\mathbf{U}'X - 2\mathbf{U}) - 2\lambda X),$$
$$\mathbf{u} = \frac{1}{2}\left(\frac{\mathbf{Y}}{2n\sqrt{\lambda}} - \mathbf{P}\right), \qquad \mathbf{z} = \frac{1}{2}\left(\frac{\mathbf{Y}}{2n\sqrt{\lambda}} + \mathbf{P}\right).$$

Finally, for later Lemmas, we set $\mathbf{w} = \mathbf{z}/\mathbf{u} \in K(\sqrt{\lambda})(X)$.

Then for $r \in \mathbb{N}$, the polynomials $\mathbf{A}_r, \mathbf{B}_r$ given by

$$(\sqrt{\lambda})^r \mathbf{A}_r = \mathbf{a}\chi^*_{n,r}(\mathbf{z}, \mathbf{u}) - \mathbf{b}\chi^*_{n,r}(\mathbf{u}, \mathbf{z}),$$
$$(\sqrt{\lambda})^r \mathbf{B}_r = \mathbf{c}\chi^*_{n,r}(\mathbf{z}, \mathbf{u}) - \mathbf{d}\chi^*_{n,r}(\mathbf{u}, \mathbf{z})$$

are elements of the polynomial ring $K[X]$ over K. For every root α of $\mathbf{P}$, the polynomial

$$\mathbf{C}_r = \alpha \mathbf{A}_r - \mathbf{B}_r$$

is divisible by $(X - \alpha)^{2r+1}$.

When constructing the approximations of Lemma A we will have to make sure that $p_r q_{r+1} \neq p_{r+1} q_r$ for all r. We will use the following lemma [9, Lemma 2.2], which is a special case of [1, Lemma 2.7].

Lemma C. *Let $\mathbf{A}_r$, $\mathbf{B}_r$, $\mathbf{P}$ and $\mathbf{U}$ be defined as in Lemma B. If $\mathbf{U}(\xi)\mathbf{P}(\xi) \neq 0$ for a given $\xi \in \mathbb{C}$, then for all positive integers r, we have*

$$\mathbf{A}_{r+1}(\xi)\mathbf{B}_r(\xi) \neq \mathbf{A}_r(\xi)\mathbf{B}_{r+1}(\xi).$$

The approximations $\mathbf{A}_r(\xi)/\mathbf{B}_r(\xi)$ will be very good; however, in general $\mathbf{A}_r(\xi), \mathbf{B}_r(\xi)$ won't be integers (even if ξ is), mainly because of the denominators coming from $\chi_{n,r}$. To clear these denominators, we will use the following lemma, which is a result by Lettl et al. [14, Proposition 2c]. In our version of the lemma we have rewritten and slightly weakened the two inequalities when rounding, to make them more naturally applicable in our context. Note that in the application in the next section, ξ will indeed have the shape $1 - 8x$ as in the lemma.

Lemma D (Lettl et al.). *Let r be a positive integer, $\Delta_{4,r}$ be the least common multiple of the denominators of the coefficients of $\chi_{4,r}$ and let $N_{4,r}$ be the greatest common divisor of the numerators of the coefficients of $\chi_{4,r}(1 - 8X)$. Then $\Delta_{4,r}/N_{4,r}\chi_{4,r}(1 - 8X)$ is a polynomial with integer coefficients and we have*

$$\frac{2^{r+2}\Delta_{4,r}}{N_{4,r}} \cdot \frac{\Gamma(3/4)r!}{\Gamma(r+3/4)} < 3.32 \cdot 1.35^r \quad \textit{and} \quad \frac{2^{4r+3}\Delta_{4,r}}{N_{4,r}} \cdot \frac{\Gamma(r+5/4)}{\Gamma(1/4)r!} < 1.6 \cdot 10.7^r.$$

With the help of the factors $\Delta_{4,r}/N_{4,r}$ from the above lemma, we will construct the actual approximations p_r/q_r to our root α. In order to apply Lemma A we will first have to estimate $|q_r|$. On the one hand, we will use the estimates from the above Lemma D, on the other hand, we will need estimates for $\chi_{4,r}$. The next lemma follows from [21, Lemma 7.3b] with $m = 1$, $n = 4$, $z = 1$ and $u = u/z = w$. Note that we cannot use [14, Lemma 4] because we won't have $|w| = 1$ since in our case t is not real.

Lemma E (Voutier). *Let w be a complex number with $|1 - w| < 1$ and $|1 - w^{-1}| < 1$ and let r be a positive integer. Then*

$$|\chi_{4,r}(w)| \leq \frac{\Gamma(3/4)2^{r+1}r!}{\Gamma(r+3/4)} \cdot (1 + |w|)^r.$$

Finally, in order to estimate $|q_r\alpha - p_r|$, which will be a multiple of $\mathbf{C}_r(\xi)$ from Lemma B, we will use the following Lemma. It is a combination of Lemmas 2.3 and 2.4 in [9]. Note that the former is proven by Chen and Voutier [1, Lemma 2.3] and that the requirement $\xi \neq 0$ was removed in a recent version of the paper, see the Addendum of [2]. For roots of complex numbers we will agree to choose the root where the argument has the smallest absolute value (and in case of ambiguity with the positive value), i.e. in the lemma below we have $-\pi/n < \arg(w(\xi)^{1/n}) \leq \pi/n$.

Lemma F (Chen and Voutier; Heuberger). *Let $n \geq 2$ and r be positive integers and let α, λ, $\mathbf{a}(X)$, $\mathbf{b}(X)$, $\mathbf{c}(X)$, $\mathbf{d}(X)$, $\mathbf{C}_r(X)$, $\mathbf{u}(X)$, $\mathbf{z}(X)$ and $\mathbf{w}(X)$ be as in Lemma B. Further, let ξ be a complex number such that $|\mathbf{w}(\xi) - 1| < 1$. Then we can write*

$$(20) \qquad \begin{aligned}(\sqrt{\lambda})^r \mathbf{C}_r(\xi) = &(\alpha(\mathbf{a}(\xi)\mathbf{w}(\xi)^{1/n} - \mathbf{b}(\xi)) - (\mathbf{c}(\xi)\mathbf{w}(\xi)^{1/n} - \mathbf{d}(\xi))) \cdot \chi^*_{n,r}(\mathbf{u}(\xi), \mathbf{z}(\xi)) \\ &- (\alpha\mathbf{a}(\xi) - \mathbf{c}(\xi)) \cdot \mathbf{u}(\xi)^r \cdot R_{n,r}(\mathbf{w}(\xi)),\end{aligned}$$

with the estimate

$$|R_{n,r}(\mathbf{w}(\xi))| \leq \frac{\Gamma(r+1+1/n)}{r!4^r\Gamma(1/n)} \cdot \frac{|\mathbf{w}(\xi)-1|^{2r+1}}{(1-|\mathbf{w}(\xi)-1|)^{r+1-1/n}}.$$

9. Proof of Proposition 1 (irrationality measure)

In this section, we prove Proposition 1, i.e. we find an effective measure of irrationality for the roots $\alpha = \alpha^{(0)}$ and $\alpha^{(3)}$ of the polynomial $f_t(X) = X^4 - tX^3 - 6X^2 + tX + 1$, with an imaginary quadratic integer t with $|t| \geq 100$.

We start by determining the quantities defined in Lemma B. First, put

$$\mathbf{P}(X) = f_t(X) = X^4 - tX^3 - 6X^2 + tX + 1.$$

Then one can check that there indeed exists a square-free quadratic polynomial $\mathbf{U}$ that satisfies the differential equation (19), namely the polynomial

$$\mathbf{U}(X) = X^2 + 1.$$

We have $\mathrm{disc}(\mathbf{U}) = -4$ and we set $\lambda = -1$, as well as

$$\begin{aligned}
\mathbf{Y}(X) &= 2tX^4 + 32X^3 - 12tX^2 - 32X + 2t, \\
\mathbf{a}(X) &= 5iX - 5, & \mathbf{c}(X) &= -5X - 5i, \\
\mathbf{b}(X) &= 5iX + 5, & \mathbf{d}(X) &= 5X - 5i, \\
\mathbf{u}(X) &= -\frac{it+4}{8}(X+i)^4, & \mathbf{z}(X) &= \frac{-it+4}{8}(X-i)^4, \\
\mathbf{w}(X) &= \frac{it-4}{it+4} \cdot \frac{(X-i)^4}{(X+i)^4}.
\end{aligned}$$

For $\mathbf{A}_r$ and $\mathbf{B}_r$ we get the formulas

$$\begin{aligned}
\mathbf{A}_r &= (-i)^r(\mathbf{a}\chi^*_{n,r}(\mathbf{z},\mathbf{u}) - \mathbf{b}\chi^*_{n,r}(\mathbf{u},\mathbf{z})) = (-i)^r(\mathbf{a}\mathbf{u}^r\chi_{n,r}(\mathbf{w}) - \mathbf{b}\mathbf{z}^r\chi_{n,r}(\mathbf{w}^{-1})), \\
\mathbf{B}_r &= (-i)^r(\mathbf{c}\chi^*_{n,r}(\mathbf{z},\mathbf{u}) - \mathbf{d}\chi^*_{n,r}(\mathbf{u},\mathbf{z})) = (-i)^r(\mathbf{c}\mathbf{u}^r\chi_{n,r}(\mathbf{w}) - \mathbf{d}\mathbf{z}^r\chi_{n,r}(\mathbf{w}^{-1})).
\end{aligned}$$

Lemma B then implies that $\mathbf{C}_r = \alpha A_r - B_r$ is divisible by $(X-\alpha)^{2r+1}$, i.e. we can expect $|\alpha A_r - B_r|$ to be very small if evaluated at a ξ close to the root α. Thus, if we want to approximate $\alpha = \alpha^{(0)}$, we have to choose a ξ close to $\alpha^{(0)}$. In view of Lemma 4, $\xi = 0$ will be a good choice for ξ. Similarly, if we want to approximate $\alpha^{(3)}$ we will choose $\xi = 1$.

9.1. Irrationality measure for $\alpha^{(0)}$

Let us first focus on $\alpha^{(0)} \approx 0$. We compute:

$$\begin{aligned}
\mathbf{a}(0) &= -5, & \mathbf{c}(0) &= -5i, \\
\mathbf{b}(0) &= 5, & \mathbf{d}(0) &= -5i, \\
\mathbf{u}(0) &= -\frac{it+4}{8} =: -\frac{u}{8}, & \mathbf{z}(0) &= \frac{-it+4}{8} =: -\frac{z}{8}, \\
\mathbf{w}(0) &= \frac{it-4}{it+4} = 1 - \frac{8}{u} := w.
\end{aligned}$$

where we defined

$$u = it + 4, \quad z = it - 4 \quad \text{and} \quad w = \frac{it-4}{it+4}.$$

Noting that $\mathbf{w}(0)^{-1} = w^{-1} = \frac{it+4}{it-4} = 1 + \frac{8}{z}$ we obtain from the definitions

$$\begin{aligned}\mathbf{A}_r(0) &= (-i)^r \left(-5\left(-\frac{u}{8}\right)^r \chi_{4,r}(1-\frac{8}{u}) - 5\left(-\frac{z}{8}\right)^r \chi_{4,r}(1+\frac{8}{z})\right)\\ &= -5i^r 8^{-r}\left(u^r\chi_{4,r}(1-\frac{8}{u}) + z^r\chi_{4,r}(1+\frac{8}{z})\right),\\ \mathbf{B}_r(0) &= (-i)^r \left(-5i\left(-\frac{u}{8}\right)^r \chi_{4,r}(1-\frac{8}{u}) + 5i\left(-\frac{z}{8}\right)^r \chi_{4,r}(1+\frac{8}{z})\right)\\ &= -5i^{r+1} 8^{-r}\left(u^r\chi_{4,r}(1-\frac{8}{u}) - z^r\chi_{4,r}(1+\frac{8}{z})\right).\end{aligned}$$

In order to obtain algebraic integers, we clear the denominators of $\mathbf{A}_r(0)$ and $\mathbf{B}_r(0)$ with the notation from Lemma D: We set

$$M_r(0) = \frac{8^r}{5}\cdot\frac{\Delta_{4,r}}{N_{4,r}} \quad\text{and}\quad p_r(0) = M_r(0)\mathbf{B}(0), \quad q_r(0) = M_r(0)\mathbf{A}(0).$$

Then we have

$$\begin{aligned}p_r(0) &= -i^{r+1}\frac{\Delta_{4,r}}{N_{4,r}}\left(u^r\chi_{4,r}(1-8u^{-1}) - z^r\chi_{4,r}(1+8z^{-1})\right),\\ q_r(0) &= -i^{r}\frac{\Delta_{4,r}}{N_{4,r}}\left(u^r\chi_{4,r}(1-8u^{-1}) + z^r\chi_{4,r}(1+8z^{-1})\right).\end{aligned}$$

We check that $p_r(0), q_r(0)$ are in $\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ for any r. First, recall that by Lemma B the polynomials $\mathbf{A}_r(X), \mathbf{B}_r(X)$ are polynomials with coefficients in $\mathbb{Q}(\sqrt{-d})$ and therefore $\mathbf{A}_r(0), \mathbf{B}_r(0) \in \mathbb{Q}(\sqrt{-d})$. Since $M_r(0) \in \mathbb{Q}$, we clearly have $p_r(0), q_r(0) \in \mathbb{Q}(\sqrt{-d})$. Now we check that $p_r(0), q_r(0)$ are algebraic integers. The factors $-i^r$ and $-i^{r+1}$ are algebraic integers. By the definition of $\Delta_{4,r}$ and $N_{4,r}$ in Lemma D we have that $\Delta_{4,r}/N_{4,r}\cdot\chi_{4,r}(1-8X)$ is a polynomial with integer coefficients of degree r. Therefore, since u, z are algebraic integers, we see that $u^r\cdot\Delta_{4,r}/N_{4,r}\cdot\chi_{4,r}(1-8u^{-1})$ and $z^r\cdot\Delta_{4,r}/N_{4,r}\cdot\chi_{4,r}(1-8(-z^{-1}))$ are algebraic integers as well. Thus $p_r(0)$ and $q_r(0)$ are algebraic integers and since they are in $\mathbb{Q}(\sqrt{-d})$, we have indeed $p_r(0), q_r(0) \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$.

In order to apply Lemma A we need to estimate $|q_r(0)|$ and $|q_r(0)\alpha - p_r(0)|$ from above. Recall that we have set

$$w = \mathbf{w}(0) = 1 - 8u^{-1} = (1+8z^{-1})^{-1} = \frac{it-4}{it+4}.$$

To check that these equalities hold, see the computation of $\mathbf{w}(0)$ and the computations below that. Now we can write

$$\begin{aligned}p_r(0) &= -i^{r+1}\frac{\Delta_{4,r}}{N_{4,r}}\left(u^r\chi_{4,r}(w) - z^r\chi_{4,r}(w^{-1})\right),\\ q_r(0) &= -i^{r}\frac{\Delta_{4,r}}{N_{4,r}}\left(u^r\chi_{4,r}(w) + z^r\chi_{4,r}(w^{-1})\right).\end{aligned}$$

In order to apply Lemma E, we check that $|1-w| < 1$ and $|1-w^{-1}| < 1$:

$$|1-w| = \left|\frac{-8}{it-4}\right| \leq \frac{8}{|t|-4} < 1 \tag{21}$$

and analogously we obtain $|1-w^{-1}|<1$. Now we use Lemma E, the fact that $|u|=|it+4|\leq |t|+4$ and $|z|=|it-4|\leq |t|+4$:

$$\begin{aligned}|q_r(0)| &\leq \frac{\Delta_{4,r}}{N_{4,r}}\left(|u|^r|\chi_{4,r}(w)|+|z|^r|\chi_{4,r}(w^{-1})|\right)\\ &\leq \frac{\Delta_{4,r}}{N_{4,r}}(|t|+4)^r\left(\frac{\Gamma(3/4)2^{r+1}r!}{\Gamma(r+3/4)}\cdot(1+|w|)^r+\frac{\Gamma(3/4)2^{r+1}r!}{\Gamma(r+3/4)}\cdot(1+|w^{-1}|)^r\right)\\ &= \frac{\Delta_{4,r}}{N_{4,r}}\frac{\Gamma(3/4)2^{r+1}r!}{\Gamma(r+3/4)}(|t|+4)^r\left((1+|w|)^r+(1+|w^{-1}|)^r\right).\end{aligned}$$

Next we use the estimates $|w|,|w^{-1}|\leq 1+8/(|t|-4)<1.09$ and $|t|+4\leq 1.04|t|$ for $|t|\geq 100$, as well as Lemma D, obtaining

$$\begin{aligned}|q_r(0)| &\leq \frac{2^{r+1}\Delta_{4,r}}{N_{4,r}}\frac{\Gamma(3/4)r!}{\Gamma(r+3/4)}(|t|+4)^r\cdot 2\cdot 2.09^r\\ &\leq \frac{2^{r+2}\Delta_{4,r}}{N_{4,r}}\frac{\Gamma(3/4)r!}{\Gamma(r+3/4)}(1.04|t|)^r\cdot 2.09^r\\ &\leq 3.32\cdot 1.35^r\cdot(1.04|t|)^r\cdot 2.09^r\\ &< 3.32\cdot(2.94|t|)^r.\end{aligned}$$

Thus we can set $k_0=3.32$ and $Q=2.94|t|$ in Lemma A. Next, we need to find an upper bound for the estimation error

$$|\alpha q_r(0)-p_r(0)|=|\alpha M_r(0)\mathbf{A}_r(0)-M_r(0)\mathbf{B}_r(0)|=M_r(0)|\mathbf{C}_r(0)|.$$

We want to apply Lemma F and we first show that the coefficient $\alpha(\mathbf{a}(0)\mathbf{w}(0)^{1/4}-\mathbf{b}(0))-(\mathbf{c}(0)\mathbf{w}(0)^{1/4}-\mathbf{d}(0))$ of $\chi^*_{4,r}(\mathbf{u}(0),\mathbf{z}(0))$ in (20) vanishes. To that aim we verify that the expression

$$\frac{\mathbf{c}(0)\mathbf{w}(0)^{1/4}-\mathbf{d}(0)}{\mathbf{a}(0)\mathbf{w}(0)^{1/4}-\mathbf{b}(0)}=i\cdot\frac{\left(\frac{it-4}{it+4}\right)^{1/4}-1}{\left(\frac{it-4}{it+4}\right)^{1/4}+1}\tag{22}$$

is a root of $f_t(X)$. This can be done with a straightforward computation. Moreover, since $(it-4)/(it+4)\approx 1$ for large $|t|$, the absolute value of the expression in (22) is very small for $|t|\geq 100$. Therefore, the above expression must be exactly $\alpha=\alpha^{(0)}$ and the first summand in (20) vanishes. Thus we obtain from Lemma F that

$$\begin{aligned}|\mathbf{C}_r(0)| &\leq |\alpha\mathbf{a}(0)-\mathbf{c}(0)|\cdot|\mathbf{u}(0)|^r\cdot\frac{\Gamma(r+1+1/4)}{r!4^r\Gamma(1/4)}\cdot\frac{|\mathbf{w}(0)-1|^{2r+1}}{(1-|\mathbf{w}(0)-1|)^{r+1-1/4}}\\ &= |-5\alpha+5i|\cdot\left(\frac{|it+4|}{8}\right)^r\cdot\frac{\Gamma(r+5/4)}{r!4^r\Gamma(1/4)}\cdot\frac{|w-1|^{2r+1}}{(1-|w-1|)^{r+3/4}}.\end{aligned}$$

Recall from (21) that $|1-w|\leq 8/(|t|-4)$, which moreover implies

$$1-|w-1|\geq 1-\frac{8}{|t|-4}=\frac{|t|-12}{|t|-4}.$$

We continue estimating $|\mathbf{C}_r(0)|$:

$$\begin{aligned}|\mathbf{C}_r(0)| &\leq 5(|\alpha|+1)2^{-5r}(|t|+4)^r\frac{\Gamma(r+5/4)}{r!\Gamma(1/4)}\cdot\left(\frac{8}{|t|-4}\right)^{2r+1}\cdot\left(\frac{|t|-4}{|t|-12}\right)^{r+3/4}\\ &= 5(|\alpha|+1)2^{r+3}(|t|+4)^r\frac{\Gamma(r+5/4)}{r!\Gamma(1/4)}(|t|-4)^{-r-1/4}(|t|-12)^{-r-3/4}.\end{aligned}$$

Now note that for $|t|\geq 100$ we have

$$|\alpha|\leq 1/|t|+5.01/|t|^3<0.02,\quad |t|+4\leq 1.04|t|,\quad |t|-4\geq 0.96|t|,\quad |t|-12\geq 0.88|t|.$$

Thus we obtain

$$\begin{aligned}|\mathbf{C}_r(0)| &< 5\cdot 1.02\cdot 2^{r+3}\frac{\Gamma(r+5/4)}{r!\Gamma(1/4)}\cdot |t|^{-r-1}1.04^r\cdot 0.96^{-r-1/4}\cdot 0.88^{-r-3/4}\\ &< 5\cdot 2^{r+3}\cdot\frac{\Gamma(r+5/4)}{r!\Gamma(1/4)}\cdot 1.14\cdot |t|^{-1}\cdot\left(\frac{1.24}{|t|}\right)^r.\end{aligned}$$

Using this estimate, the definition of $M_r(0)$ and Lemma D we obtain

$$\begin{aligned}M_r(0)|\mathbf{C}_r(0)| &< \frac{8^r}{5}\cdot\frac{\Delta_{4,r}}{N_{4,r}}\cdot 5\cdot 2^{r+3}\cdot\frac{\Gamma(r+5/4)}{r!\Gamma(1/4)}\cdot 1.14\cdot|t|^{-1}\cdot\left(\frac{1.24}{|t|}\right)^r\\ &= \frac{2^{4r+3}\Delta_{4,r}}{N_{4,r}}\cdot\frac{\Gamma(r+5/4)}{\Gamma(1/4)r!}\cdot 1.14\cdot|t|^{-1}\cdot\left(\frac{1.24}{|t|}\right)^r\\ &< 1.6\cdot 10.7^r\cdot 1.14\cdot|t|^{-1}\cdot\left(\frac{1.24}{|t|}\right)^r\\ &< 1.83|t|^{-1}\cdot\left(\frac{13.27}{|t|}\right)^r.\end{aligned}$$

Thus we have proven that $|\alpha q_r(0)-p_r(0)| < 1.83|t|^{-1}\cdot(|t|/13.27)^{-r}$ and we can set $l_0 = 1.83/|t|$ and $E = |t|/13.27$ in Lemma A. Let us sum up what we have achieved so far: Assuming $|t|\geq 100$, for the root α we have found $p_r(0), q_r(0)\in\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ for all positive integers r, such that $|q_r(0)| < k_0Q^r$ and $|\alpha q_r(0)-p_r(0)|\leq l_0E^{-r}$ with

$$k_0 = 3.32,\quad Q = 2.94|t|,\quad l_0 = 1.83/|t|,\quad E = |t|/13.27.$$

Now we only need to check that $p_r(0)q_{r+1}(0)\neq p_{r+1}(0)q_r(0)$ for all r. This follows immediately from Lemma C as $\mathbf{U}(0)=\mathbf{P}(0)=1\neq 0$, and the fact that $M_r(0)\neq 0$ for all r. Thus we can apply Lemma A with

$$\kappa = \frac{\log Q}{\log E} = \frac{\log|t|+\log 2.94}{\log|t|-\log 13.27}\leq\frac{\log|t|+1.08}{\log|t|-2.59}.$$

Note that $\kappa < 3$ for $|t|\geq 84$. Moreover, we have

$$\begin{aligned}c = 2k_0Q(2l_0E)^\kappa &= 2\cdot 3.32\cdot 2.94|t|(2\cdot 1.83/|t|\cdot|t|/13.27)^\kappa\\ &< 19.53|t|\cdot 0.28^\kappa < 19.53|t|\cdot 0.28 < 5.47|t|.\end{aligned}$$

Finally, note that

$$1/(2l_0) = 1/(2\cdot 1.83/|t|) < 0.28|t|.$$

Thus Lemma A yields

$$\left|\alpha-\frac{p}{q}\right| > \frac{1}{5.47|t|\cdot|q|^{\kappa+1}}\quad\text{with}\quad\kappa = \frac{\log|t|+1.08}{\log|t|-2.59}$$

for all $|q|\geq 0.28|t|$.

We have proven Proposition 1 for $j=0$. Note that the worse constant 15.48 instead of 5.47 will come from the type 3 case.

9.2. Irrationality measure for $\alpha^{(3)}$

Now we quickly repeat all computations for $\alpha^{(3)}\approx 1$.

$$\mathbf{a}(1) = 5i-5,\qquad \mathbf{c}(1) = -5i-5,$$

$$\mathbf{b}(1) = 5i+5,\qquad \mathbf{d}(1) = -5i+5,$$

$$\mathbf{u}(1) = \frac{it+4}{2} = \frac{u}{2},\qquad \mathbf{z}(1) = \frac{it-4}{2} = \frac{z}{2},$$

$$\mathbf{w}(1) = \frac{it-4}{it+4} = 1-\frac{8}{u} = w,$$

where as before

$$u = it + 4, \quad z = it - 4 \quad \text{and} \quad w = \frac{it-4}{it+4} = 1 - \frac{8}{u} = \left(1 + \frac{8}{z}\right)^{-1}.$$

We continue with the computations as above:

$$\begin{aligned}
\mathbf{A}_r(1) &= (-i)^r \left((5i-5)\left(\frac{u}{2}\right)^r \chi_{n,r}(1-\frac{8}{u}) - (5i+5)\left(\frac{z}{2}\right)^r \chi_{n,r}(1+\frac{8}{z})\right) \\
&= 5(i-1)(-i)^r 2^{-r}\left(u^r \chi_{n,r}(1-\frac{8}{u}) + i\cdot z^r \chi_{n,r}(1+\frac{8}{z})\right), \\
\mathbf{B}_r(1) &= (-i)^r \left((-5i-5)\left(\frac{u}{2}\right)^r \chi_{n,r}(1-\frac{8}{u}) - (-5i+5)\left(\frac{z}{2}\right)^r \chi_{n,r}(1+\frac{8}{z})\right) \\
&= -5(i+1)(-i)^r 2^{-r}\left(u^r \chi_{n,r}(1-\frac{8}{u}) - i\cdot z^r \chi_{n,r}(1+\frac{8}{z})\right).
\end{aligned}$$

We set

$$M_r(1) = \frac{2^r}{5}\cdot\frac{\Delta_{4,r}}{N_{4,r}} \quad \text{and} \quad p_r(1) = M_r(1)\mathbf{B}(1), \quad q_r(1) = M_r(1)\mathbf{A}(1)$$

and obtain

$$\begin{aligned}
p_r(1) &= -(i+1)(-i)^r \frac{\Delta_{4,r}}{N_{4,r}}\left(u^r \chi_{4,r}(1-8u^{-1}) - i\cdot z^r \chi_{4,r}(1+8z^{-1})\right), \\
q_r(1) &= (i-1)(-i)^r \frac{\Delta_{4,r}}{N_{4,r}}\left(u^r \chi_{4,r}(1-8u^{-1}) + i\cdot z^r \chi_{4,r}(1+8z^{-1})\right).
\end{aligned}$$

The numbers $p_r(1), q_r(1)$ are in $\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ by the same arguments as for $p_r(0), q_r(0)$ above.

Next, we estimate $|q_r(1)|$. This is completely analogous to the estimate for $|q_r(0)|$ from above, except that we now have the additional factor $(i-1)$ with absolute value $\sqrt{2}$. Thus we end up with

$$|q_r(0)| < \sqrt{2}\cdot 3.32\cdot(2.94|t|)^r < 4.7\cdot(2.94|t|)^r.$$

Next, we find an upper bound for

$$|\alpha q_r(1) - p_r(1)| = |\alpha M_r(1)\mathbf{A}_r(1) - M_r(1)\mathbf{B}_r(1)| = M_r(1)|\mathbf{C}_r(1)|.$$

As above, one can check that

$$\frac{\mathbf{c}(1)\mathbf{w}(1)^{1/4} - \mathbf{d}(1)}{\mathbf{a}(1)\mathbf{w}(1)^{1/4} - \mathbf{b}(1)} = \frac{\left(\frac{it-4}{it+4}\right)^{1/4} - i}{-i\left(\frac{it-4}{it+4}\right)^{1/4} + 1}$$

is a root of $f_t(X)$, which is close to 1, and therefore must be equal to $\alpha^{(3)}$. Thus the first summand in (20) vanishes and Lemma F yields

$$\begin{aligned}
|\mathbf{C}_r(1)| &\le |\alpha^{(3)}\mathbf{a}(1) - \mathbf{c}(1)|\cdot|\mathbf{u}(1)|^r\cdot\frac{\Gamma(r+1+1/4)}{r!4^r\Gamma(1/4)}\cdot\frac{|\mathbf{w}(1)-1|^{2r+1}}{(1-|\mathbf{w}(1)-1|)^{r+1-1/4}} \\
&= |5(i-1)(\alpha^{(3)}-i)|\cdot\left(\frac{|it+4|}{2}\right)^r\cdot\frac{\Gamma(r+5/4)}{r!4^r\Gamma(1/4)}\cdot\frac{|w-1|^{2r+1}}{(1-|w-1|)^{r+3/4}}.
\end{aligned}$$

The rest of the estimation is completely analogous to that of $\mathbf{C}_r(1)$, except we now have the extra factor $|i-1| = \sqrt{2}$. Moreover, instead of the factor from before $|-\alpha+i| \le |\alpha|+1 \le 1.02$, we now have $|\alpha^{(3)} - i| \le |1-i| + 2.16/|t| \le \sqrt{2} + 2.16/100 \le 1.44$. Thus we end up with

$$\begin{aligned}
M_r(1)|\mathbf{C}_r(1)| &\le \frac{1.83\cdot\sqrt{2}\cdot 1.44}{1.02}\cdot|t|^{-1}\cdot\left(\frac{13.27}{|t|}\right)^r \\
&\le 3.66|t|^{-1}\cdot\left(\frac{13.27}{|t|}\right)^r.
\end{aligned}$$

Now we can set

$$k_0 = 4.7, \quad Q = 2.94|t|, \quad l_0 = 3.66/|t|, \quad E = |t|/13.27$$

and we get the same κ as before. For the constant c we now have

$$\begin{aligned} c = 2k_0Q(2l_0E)^\kappa &= 2 \cdot 4.7 \cdot 2.94|t|(2 \cdot 3.66/|t| \cdot |t|/13.27)^\kappa \\ &\leq 27.64|t|0.56^\kappa \leq 27.64|t|0.56 \leq 15.48|t|. \end{aligned}$$

Finally, note that

$$1/(2l_0) = 1/(2 \cdot 3.66/|t|) \leq 0.14|t|.$$

Lemma A finally yields

$$\left|\alpha - \frac{p}{q}\right| > \frac{1}{15.48|t| \cdot |q|^{\kappa+1}} \quad \text{with} \quad \kappa = \frac{\log|t| + 1.08}{\log|t| - 2.59}$$

for all $|q| \geq 0.14|t|$.

We have thus proven Proposition 1 for $i = 3$. Note that the stronger assumption $|q| \geq 0.28$ came from the case $i = 0$. Overall, Proposition 1 is now proven.

10. Proof of Theorem 1 (resolution of equation)

Since we have already solved $F_t(X, Y) = 0$ in Section 3, in order to finish the proof of Theorem 1 we only need to solve the equations of the shape

$$F_t(X, Y) = \mu. \tag{23}$$

Let $t \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ with $|t| \geq 100$, let $\mu \in \mathbb{Z}^\times_{\mathbb{Q}(\sqrt{-d})}$ and let $(x, y) \in \mathbb{Z}^2_{\mathbb{Q}(\sqrt{-d})}$ be non-trivial solution to equation (23). In view of Lemma 3 we may assume $\min\{|x|, |y|\} \geq 3$. By Lemma 6 we may assume without loss of generality that (x, y) is either of type 0 or of type 3. From Lemma 7 we get that $|y| \geq 0.44|t| \geq 0.28|t|$. Now we can combine Proposition 1 and Lemma 5:

$$\frac{1}{15.48|t| \cdot |y|^{\kappa+1}} < \left|\alpha - \frac{x}{y}\right| < \frac{8.86}{|t| \cdot |y|^4}.$$

This implies

$$|y|^{3-\kappa} < 8.86 \cdot 15.48 < 137.16$$

and with $\kappa < 2.83$

$$|y| < 137.16^{1/0.17} < 3.74 \cdot 10^{12}.$$

This contradicts the lower bound $|y| > 1.047 \cdot 10^{13}$ from Lemma 7. Thus we have proven that there are no non-trivial solutions to equation (23) for $|t| \geq 100$.

Remark 3. The proof of Theorem 1 works in principle as long as $\kappa < 3$, we just have to increase the lower bound for $|y|$ by pushing the Padé approximations in the proof of Lemma 7 further. Since $\kappa < 3$ for $|t| \geq 84$, one could extend Theorem 1 to roughly $|t| \geq 84$ with the method used in this paper. Moreover, one could slightly improve the result by estimating more carefully in the proof of Proposition 1. We refrained from this in favor of readability.

11. Proof of Corollaries 1 and 2

Corollaries 1 and 2 are concerned with the inequalities $|F_t(X, Y)| \leq C|t|$ and $|F_t(X, Y)| \leq |t|^{2-\varepsilon}$ respectively. As mentioned in Section 2, the Corollaries follow relatively quickly from Proposition 1 and in contrast to Theorem 1 the proofs make use of the fact that κ can get arbitrarily close to 1.

Before proving the Corollaries, we first generalize Lemma 5, Lemma 6 and a partial result from Lemma 7 to more general inequalities of the shape $|F_t(X, Y)| \leq Q$.

Lemma 5*. *Let $|t| \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$, $|t| \geq 100$ and $Q \in \mathbb{R}^+$. Let $(x, y) \in \mathbb{Z}^2_{\mathbb{Q}(\sqrt{-d})}$ with $y \neq 0$ be a solution to the inequality*

$$|F_t(X, Y)| \leq Q.$$

If (x, y) is of type j (i.e. $|\beta^{(j)}|$ is minimal), then

$$|x - \alpha^{(j)}y| = |\beta^{(j)}| < \frac{8.86 \cdot Q}{|t| \cdot |y|^3}.$$

Proof. The proof is completely analogous to the proof of Lemma 5. □

Lemma 6*. *Let $t \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$, $|t| \geq 100$ and $Q \in \mathbb{R}^+$. Let $(x, y) \in \mathbb{Z}^2_{\mathbb{Q}(\sqrt{-d})}$ be a solution to the inequality*

$$|F_t(X, Y)| \leq Q, \tag{24}$$

with $\min\{|x|, |y|\} \geq \left(\frac{20.14Q}{|t|}\right)^{1/4}$. Then (x, y) is of type j if and only if $(-y, x)$ is a solution of type $j + 2 \pmod 4$.

Proof. First, let us define approximations to the roots of $f_t(X)$:

$$\xi^{(0)} = 0, \quad \xi^{(1)} = -1, \quad \xi^{(2)} = t, \quad \xi^{(3)} = 1.$$

Then by Lemma 4 we have that $|\alpha^{(i)} - \xi^{(i)}| < 0.06$ for $i = 1, 2, 3, 4$.

Let $(x, y) \in \mathbb{Z}^2_{\mathbb{Q}(\sqrt{-d})}$ be any solution to (24) with $xy \neq 0$ of any type j. Then by Lemma 5* we have that

$$\left|\alpha^{(j)} - \frac{x}{y}\right| < \frac{8.86 \cdot Q}{|t| \cdot |y|^4}.$$

Now note that the assumption

$$|y| \geq \min\{|x|, |y|\} > \left(\frac{20.14Q}{|t|}\right)^{1/4} > \left(\frac{8.86Q}{0.42|t|}\right)^{1/4}$$

was chosen such that

$$\left|\alpha^{(j)} - \frac{x}{y}\right| < \frac{8.86 \cdot Q}{|t| \cdot |y|^4} < 0.44$$

In particular, this implies

$$\left|\xi^{(j)} - \frac{x}{y}\right| < 0.5. \tag{25}$$

Since the distance between any two distinct $\xi^{(i)}$ is at least 0.5, we can say that (x, y) is of type j if and only if it satisfies (25).

Now consider $(-y, x)$, which is also a solution to (24) therefore of some type k, i.e.

$$\left|\xi^{(k)} - \frac{-y}{x}\right| < 0.5.$$

Thus we have a complex number $z = x/y$ which satisfies both $|z - \xi^{(j)}| < 0.5$ and $|-z^{-1} - \xi^{(k)}| < 0.5$. Looking at the set $\{\xi^{(0)} = 0, \xi^{(1)} = -1, \xi^{(2)} = t, \xi^{(3)} = 1\}$ it is easy to see that this is only possible if either $\{j, k\} = 0, 2$ or $\{j, k\} = 1, 3$. □

Remark 4. The argument in the proof of Lemma 6* could also have been used to prove Lemma 6 without the assumption $\min\{|x|, |y|\} \geq 3$. However, the lower bound 3 was also helpful for establishing the lower bounds for $|y|$ in Section 7. In particular, we used $|y| \geq 3$ to get a sufficiently large bound of the shape $|y| > c \cdot |t|$ in Step *1* in Section 7.2. Moreover, it was interesting to see some small solutions in Section 4.

Finally, as a last preparation for the proofs of the Corollaries, we prove the following lemma. It is analogous to a partial result in the proof the of Lemma 7, which gave us lower bounds for $|y|$. Indeed, we will later use Lemma 7* to obtain lower bounds for $|y|$.

Lemma 7*. *Let,* $t \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$, $|t| \geq 100$ *and* $Q \in R^+$. *Let* $(x, y) \in \mathbb{Z}^2_{\mathbb{Q}(\sqrt{-d})}$ *with* $x \neq y \neq 0$ *be a solution to the inequality*

$$|F_t(X, Y)| \leq Q.$$

Assume that (x, y) *is either of type 0 or 3. Then we have*

$$1 < \frac{2.16}{|t|}|y| + \frac{8.86Q}{|t| \cdot |y|^3}.$$

Proof. We combine Lemma 4 and Lemma 5* in the same way as in Step *1* in Section 7.1 and in Step *1* in Section 7.2. In the type 0 case we obtain

$$1 \leq |x| < \frac{1.01}{|t|}|y| + \frac{8.86Q}{|t| \cdot |y|^3}$$

and in the type 3 case we obtain

$$1 \leq |x - y| < \frac{2.16}{|t|}|y| + \frac{8.86Q}{|t| \cdot |y|^3}.$$

Overall, we have proven the Lemma. □

Proof of Corollary 1. Let $C > 0$ be given.

First, we choose a constant t_0 such that $\kappa(t_0) < 2$. This works for any $t_0 \geq 524$, however, if t_0 is close to 524, we will have to choose C_0 extremely large. In fact, we choose C_0 in the following way:

$$C_0 = \max\{(20.14C)^{1/4}, 3C^{1/3}, (443C)^{1/(2-\kappa(t_0))}\}.$$

The motivation for this choice will become apparent later in the proof.

To prove Corollary 1, we need to check the following statement: For any square-free integer d and any $t \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ with $|t| \geq t_0$, the inequality

$$|F_t(X, Y)| \leq C|t| \quad \text{in } X, Y \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})} \tag{26}$$

has no solutions (x, y) with $\min\{|x|, |y|\} \geq C_0$, except solutions of the shape $(x, \pm x)$ with $|x| \leq (C|t|/4)^{1/4}$.

Let (x, y) be a solution to (26) with $\min\{|x|, |y|\} \geq C_0$, for some t with $|t| \geq t_0$.

First, assume that $y = \pm x$. Then we get that $|F_t(x, y)| = 4|x|^4 \leq C|t|$, which implies $|x| \leq (C|t|/4)^{1/4}$.

From now on, assume that $y \neq \pm x$ (this will be necessary for the application of Lemma 7* later). In order to use Lemma 6*, we need to check that

$$\min\{|x|, |y|\} \geq \left(\frac{20.14Q}{|t|}\right)^{1/4} = (20.14C)^{1/4}.$$

This is indeed guaranteed by $\min\{|x|, |y|\} \geq C_0 \geq (20.14C)^{1/4}$. Thus by Lemma 6* we may assume without loss of generality that (x, y) is either of type 0 or of type 3.

Next, we can use Lemma 7*, which (with $Q = C|t|$) gives us

$$1 < \frac{2.16}{|t|}|y| + \frac{8.86C}{|y|^3}.$$

Since we are assuming $|y| \geq C_0 \geq 3C^{1/3}$, we obtain

$$1 < \frac{2.16}{|t|}|y| + 0.33, \tag{27}$$

which implies

$$|y| > 0.31|t|. \tag{28}$$

In particular, we have $|y| \geq 0.28|t|$, and we can apply Proposition 1 and combine it with Lemma 5* (with $Q = C|t|$):

$$\frac{1}{15.48|t||y|^{\kappa+1}} < \left|\alpha^{(j)} - \frac{x}{y}\right| < \frac{8.86 \cdot C \cdot |t|}{|t| \cdot |y|^4}.$$

This implies

$$|y|^{3-\kappa} < C_2|t|,$$

where $C_2 = 8.86 \cdot C \cdot 15.48$. Combining this with (28) we obtain

$$|y|^{3-\kappa} < C_3|y|,$$

with $C_3 = 443C > C_2/0.31$. Now the inequality $|y|^{3-\kappa} < 443C|y|$ implies $|y|^{2-\kappa} < 443C$ and thus

$$|y| < (443C)^{1/(2-\kappa)}. \tag{29}$$

Assume for a moment that $443C < 1$. Then $|y| < 1$, which implies $y = 0$ and is excluded because of $C_0 > 0$. Thus we may assume that $443C \geq 1$. Since $\kappa = \kappa(t) \leq \kappa(t_0)$ for all $t \geq t_0$, inequality (29) then implies $|y| < (443C)^{1/(2-\kappa(t_0))}$. This contradicts our assumption $|y| \geq C_0 \geq (443C)^{1/(2-\kappa(t_0))}$. □

Proof of Corollary 2. Let $0 < \varepsilon < 1$. We need prove that there exists an effectively computable constant $t_0 \geq 100$ such that the following statement holds: For any square-free integer d and any $t \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ with $|t| \geq t_0$ the inequality

$$|F_t(X,Y)| \leq |t|^{2-\varepsilon} \quad \text{in } X,Y \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})} \tag{30}$$

has no solutions (x,y) with $\min\{|x|,|y|\} > (|t|^{2-\varepsilon}/4)^{1/4}$.

Assume that there exists a solution (x,y) to (30) with $\min\{|x|,|y|\} > (|t|^{2-\varepsilon}/4)^{1/4}$.

If $y = \pm x$. Then we get that $|F_t(x,y)| = 4|x|^4 \leq |t|^{2-\varepsilon}$, which implies $|x| \leq (|t|^{2-\varepsilon}/4)^{1/4}$, a contradiction. From now on, assume that $y \neq \pm x$ (this will be necessary for the application of Lemma 7* later).

In order to use Lemma 6*, we need to check that

$$\min\{|x|,|y|\} \geq \left(\frac{20.14Q}{|t|}\right)^{1/4} = (20.14|t|^{1-\varepsilon})^{1/4} = 20.14^{(1-\varepsilon)/4} \cdot |t|^{1/4-\varepsilon/4}.$$

This is indeed guaranteed by $\min\{|x|,|y|\} > |t|^{1/2-\varepsilon/4}$, if $|t|$ is large enough. Thus by Lemma 6* we may assume without loss of generality that (x,y) is either of type 0 or of type 3.

Next, we can use Lemma 7* with $Q = |t|^{2-\varepsilon}$ and we get

$$1 \leq \frac{2.16|y|}{|t|} + \frac{8.86|t|^{1-\varepsilon}}{|y|^3}.$$

Using $|y| > |t|^{1/2-\varepsilon/4}$, we obtain

$$1 < \frac{2.16|y|}{|t|} + \frac{8.86|t|^{1-\varepsilon}}{|t|^{3/2-3\varepsilon/4}} = \frac{2.16|y|}{|t|} + \frac{8.86}{|t|^{1/2+\varepsilon/4}}.$$

If $|t|$ is large enough, the last summand is at most 0.33. Thus, as in the previous proof, we end up with inequality (27), which implies

$$|y| > 0.31|t|. \tag{31}$$

In particular, we have $|y| \geq 0.28|t|$, and we can apply Proposition 1 and combine it with Lemma 5*:

$$\frac{1}{15.48|t||y|^{\kappa+1}} < \left|\alpha^{(j)} - \frac{x}{y}\right| < \frac{8.86|t|^{1-\varepsilon}}{|y|^4}.$$

This implies

$$|y|^{3-\kappa} < 137.16|t|^{2-\varepsilon}.$$

Combining that with (31), we obtain

$$|y|^{3-\kappa} < 137.16/0.31^{2-\varepsilon} \cdot |y|^{2-\varepsilon},$$

which implies

$$|y| < (137.16/0.31^{2-\varepsilon})^{1/(1+\varepsilon-\kappa(t))} < (|t|^{2-\varepsilon}/4)^{1/4},$$

a contradiction.

Note that the last inequality holds if $|t|$ is large enough, since $\kappa(t)$ gets arbitrarily close to 1 if $|t|$ is large enough and $(|t|^{2-\varepsilon}/4)^{1/4}$ grows as $|t|$ grows. In all the arguments of this proof "large enough" was always effectively computable in terms of ε. Thus we have proven Corollary 2. □

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Appendix

Let $m > 0$ be some given bound. We describe how to give a list of all imaginary quadratic integers $0 < |x| \le m$ with either $\Im(x) > 0$ or $0 < x \in \mathbb{Z}$ (i.e. up to sign the full list of quadratic integers in the given range).

Any quadratic integer $x \in \mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}$ can be written as

$$x = a + b\omega \quad \text{with} \quad a, b \in \mathbb{Z} \quad \text{and} \quad \omega = \begin{cases} \frac{1+\sqrt{-d}}{2}, & \text{if} \quad -d \equiv 1 \pmod 4, \\ \sqrt{-d}, & \text{else.} \end{cases}$$

Then we have

$$|x|^2 = \begin{cases} (a + \frac{b}{2})^2 + d(\frac{b}{2})^2, & \text{if} \quad -d \equiv 1 \pmod 4, \\ a^2 + db^2, & \text{else.} \end{cases} \tag{32}$$

Therefore, we only need to check d's with the following properties: $d \ge 1$, d is square free and $1/4 + d/4 \le m^2$ if $d \equiv 1 \pmod 4$ and $d \le m^2$ else. For example, for $m = 3$, we get that

$$d \in \{1, 2, 3, 5, 6, 7, 11, 15, 19, 23, 31, 35\}.$$

Then for each d we need to find all $a, b \in \mathbb{Z}$ such that $|a + b\omega| \le m$. If we first add all integers $1, 2, \dots, \lfloor m \rfloor$ to our list, we can assume that $b \ge 1$. From (32) we get the following bounds: If $-d \equiv 1 \pmod 4$, then

$$1 \le b \le 2m/\sqrt{d} \quad \text{and} \quad -\sqrt{m^2 - d\left(\frac{b}{2}\right)^2} - \frac{b}{2} \le a \le \sqrt{m^2 - d\left(\frac{b}{2}\right)^2} - \frac{b}{2}.$$

If $-d \equiv 2, 3 \pmod 4$, then

$$1 \le b \le m/\sqrt{d} \quad \text{and} \quad -\sqrt{m^2 - db^2} \le a \le \sqrt{m^2 - db^2}.$$

Then we only need to loop through all such d's, b's and a's.

For example, for $m = 3$ we obtain a list of 76 quadratic integers; here is a very short summary:

$$x \in \{1, 2, 3, \pm 2 + i, \pm 1 + i, i, 2i \pm 2, \dots, \frac{\pm 1 + \sqrt{-35}}{2}\}.$$

Finally, in order to not rely on hyperlinks, here are the links to the Sage code, that has been referred to in the paper:

`https://cocalc.com/IngridVukusic/QuarticFamily/Irreducibility` for the proof of Lemma 1,

`https://cocalc.com/IngridVukusic/QuarticFamily/SmallSolutions` for the proof of Lemma 3,

`https://cocalc.com/IngridVukusic/QuarticFamily/Rouche` for the proofs of Lemma 4, 8 and 9, https://cocalc.com/IngridVukusic/QuarticFamily/LowerBoundForY for the proof of Lemma 7.

B. Earp-Lynch, Carleton University, 1125 Colonel By Dr, Ottawa, ON, Canada
Email address: benjaminearplynch@cmail.carleton.ca

B. Faye, UFR SATIC, Université Alioune Diop de Bambey, Diourbel, Bambey 30, Sénégal
Email address: bernadette.faye@uadb.edu.sn

E. Goedhart, Franklin & Marshall College, 415 Harrisburg Ave, Lancaster, PA 17603, United States of America
Email address: eva.goedhart@fandm.edu

I. Vukusic, University of Salzburg, Hellbrunnerstrasse 34/I, 5020 Salzburg, Austria
Email address: ingrid.vukusic@plus.ac.at

D. P. Wisniewski, Department of Mathematics/Computer Science, DeSales University, Center Valley, PA 18034, United States of America
Email address: daniel.wisniewski@desales.edu

THUE EQUATIONS OVER $\mathbb{C}(T)$: THE COMPLETE SOLUTION OF A SIMPLE QUARTIC FAMILY

BERNADETTE FAYE, INGRID VUKUSIC, EZRA WAXMAN, AND VOLKER ZIEGLER

Abstract. In this paper we completely solve a simple quartic family of Thue equations over $\mathbb{C}(T)$. Specifically, we apply the ABC-Theorem to find all solutions $(x, y) \in \mathbb{C}[T]\times\mathbb{C}[T]$ to the set of Thue equations $F_\lambda(X, Y) = \xi$, where $\xi \in \mathbb{C}^\times$ and

$$F_\lambda(X, Y) := X^4 - \lambda X^3Y - 6X^2Y^2 + \lambda XY^3 + Y^4, \qquad \lambda \in \mathbb{C}[T]/\{\mathbb{C}\}$$

denotes a family of quartic simple forms.

1. Introduction

Diophantine equations, named after Diophantus of Alexandra, have been an enduring topic of mathematical interest from antiquity up until the modern era. Pythagoras, for example, studied integer solutions to the equation $X^2 + Y^2 = Z^2$, while Brahmagupta, Euler, and Fermat studied such solutions to the equation $61X^2 + 1 = Y^2$. By the twentieth century, a much richer general theory of Diophantine equations began to emerge. Axel Thue [20], for instance, considered equations of the form $F(X, Y) = m$, where m is a non-zero integer, and $F(X, Y) \in \mathbb{Z}[X, Y]$ is an irreducible homogeneous *binary form* of degree $n \geq 3$. In 1909, he managed to prove that such equations (now known as *Thue equations*) have only finitely many integer solutions $(x, y) \in \mathbb{Z}^2$. Thue's result, however, was not *effective*, i.e. did not provide a bound for the size of such solutions. Baker [1] resolved this in the 1960's, by developing powerful methods to compute lower bounds for linear forms in logarithms. Such tools could then be applied to solve Thue equations effectively. In other words, Baker's method managed to reduce, to a finite amount of computation, the problem of determining all integer solutions $(x, y) \in \mathbb{Z}^2$ to a given Thue equation.

1.1. **Families of Thue Equations.** One direction of investigation then turned towards studying parametrized *families* of Thue equations. E. Thomas [19], for instance, considered the family of cubic forms

$$F_t^{(3)}(X, Y) := X^3 - (t - 1)X^2Y - (t + 2)XY^2 - Y^3 \tag{1}$$

for $t \in \mathbb{Z}_{\geq 0}$. He conjectured that for $t \geq 4$, the Thue equation

$$F_t^{(3)}(X, Y) = \pm 1$$

has only the "trivial" solutions $(x, y) \in \{(0, \mp 1), (\pm 1, 0), (\mp 1, \pm 1)\}$. Such a conjecture was eventually proved correct by Mignotte [14]. More general questions related to such Thue equations were addressed in [6, 10]. Lettl and Pethő [9] then investigated the family of quartic forms

$$F_t^4(X, Y) := X^4 - tX^3Y - 6X^2Y^2 + tXY^3 + Y^4 \tag{2}$$

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and determined the complete solution set for Thue equations of the form $F_t^4(X,Y) = m$, where $t \in \mathbb{Z}$ and $m \in \{\pm 1, \pm 4\}$. The families in (1) and (2) are known as *simple forms*, and are discussed below in Section 1.3 in further detail. For a general survey discussion about families of Thue equations see [8].

1.2. **Thue Equations Over Function Fields.** One may also consider Thue equations in the function field setting. More precisely, we consider equations of the form $F(X,Y) = m$, for some non-zero $m \in \mathbb{C}[T]$, where

$$F(X,Y) = a_0X^n + a_1X^{n-1}Y + \cdots + a_{n-1}XY^{n-1} + a_nY^n, \qquad a_i \in \mathbb{C}[T],$$

is irreducible, and where we now seek solutions $(x,y) \in \mathbb{C}[T] \times \mathbb{C}[T]$. By applying a function field analogue of Thue's method, Gill [7] demonstrated that the solutions to any such equation have bounded degree. Using methods developed by Osgood [16], Schmidt managed to obtain explicit bounds on the degree of such solutions. In contrast to classical Thue equations, however, such a bound does not directly imply that only finitely many such solutions exist. Mason [11, 12] eventually succeeded in demonstrating that the solution set of a Thue equation over $\mathbb{C}(T)$ may be effectively determined. For a history on the development of Thue equations over function fields see [13].

Families of Thue equations over $\mathbb{C}(T)$ were first discussed in [4], and the $\mathbb{C}(T)$ analogue of (1) was resolved in [5]. The purpose of this work is to investigate the $\mathbb{C}(T)$ analogue of (2). We obtain the following result:

Theorem 1. *Fix a non-constant $\lambda \in \mathbb{C}[T]$, and consider the (homogeneous) polynomial*

$$F_\lambda(X,Y) := X^4 - \lambda X^3Y - 6X^2Y^2 + \lambda XY^3 + Y^4. \tag{3}$$

Then for any $\xi \in \mathbb{C}^\times$ the solution set of the Thue equation

$$F_\lambda(X,Y) = \xi$$

is equal to

$$\begin{aligned}\mathcal{S}_{\lambda,\xi} &:= \{(x,y) \in \mathbb{C}[T] \times \mathbb{C}[T] : F_\lambda(x,y) = \xi\} \\ &= \{(\eta,0),(0,\eta) : \eta^4 = \xi\} \cup \{(\eta,\eta),(\eta,-\eta) : -4\eta^4 = \xi\}.\end{aligned}$$

1.3. **Simple Forms.** To motivate the study of simple forms, consider the *Möbius* map $\phi : z \mapsto \frac{az+b}{cz+d}$, with $a,b,c,d \in \mathbb{Z}$. Let $G_\phi = \langle \phi \rangle$ denote the cyclic group generated by ϕ. If ϕ has finite order, it may be shown that $|G_\phi| \in \{1,2,3,4,6\}$. Let ϕ be a *Möbius* map of finite order, and suppose there exists an irreducible form $F(X,Y) \in \mathbb{Z}[X,Y]$ of degree $n \in \{3,4,6\}$ such that G_ϕ acts transitively on the roots of $F(X,1)$. Lettl, Pethő, and Voutier [10] refer to such forms as *simple forms*.

As an example, consider the map $\phi : z \mapsto \frac{-1}{z+1}$, which generates a cyclic group G_ϕ of order 3. We ask for the set of irreducible cubic polynomials $f(X)$ upon whose roots G_ϕ acts transitively. Such polynomials must be of the form

$$\begin{aligned}f_t^{(3)}(X) &= (X-\alpha)(X-\phi(\alpha))(X-\phi^2(\alpha)) \\ &= X^3 + \left(\frac{1}{\alpha} + \frac{1}{1+\alpha} - \alpha + 1\right)X^2 + \left(\frac{1}{\alpha} + \frac{1}{1+\alpha} - \alpha - 2\right)X - 1 \\ &= X^3 - (t-1)X^2 - (t+2)X - 1\end{aligned}$$

where α denotes a root of $f_t^{(3)}(X)$, and where $t := \alpha - \frac{1}{\alpha} - \frac{1}{1+\alpha}$. We then obtain the family of simple cubic forms in (1) upon restricting $t \in \mathbb{Z}_{\geq 0}$.

Two forms $F(X,Y), G(X,Y) \in \mathbb{Q}[X,Y]$ are said to be *equivalent* if there exists a $t \in \mathbb{Q}^\times$ and a matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in GL_2(\mathbb{Q})$ such that $G(X,Y) = t \cdot F(pX+qY, rX+sY)$. It may be demonstrated that any simple form is equivalent to a form in one of the following two parameter families:

$$F^{(3)}_{s,t}(X,Y) = sX^3 - (t-s)X^2Y - (t+2s)XY^2 - sY^3,$$
$$F^{(4)}_{s,t}(X,Y) = sX^4 - tX^3Y - 6sX^2Y^2 + tXY^3 + sY^4,$$
$$F^{(6)}_{s,t}(X,Y) = sX^6 - 2tX^5Y - (5t+15s)X^4Y^2 - 20sX^3Y^3 + 5tX^2Y^4 + (2t+6s)XY^5 + sY^6.$$

Above we only consider irreducible such forms, and moreover restrict $s \in \mathbb{N}$, $t \in \mathbb{Z}$ such that $(s,t) = 1$. These two-parameter families of forms have been studied in [21] by applying the hypergeometric method.

When $s = 1$, the corresponding polynomial $f^{(i)}_t(X) := F^{(i)}_{1,t}(X,1)$ is monic with constant term ± 1, which enables an easier application of Baker's method to the study of such forms. Note that the family of cubic forms $F^{(3)}_{1,t}(X,Y)$, $t \in \mathbb{Z}_{\geq 0}$, corresponds to those in (1), while the family of quartic forms $F^{(3)}_{1,t}(X,Y)$, $t \in \mathbb{Z}$, corresponds to those in (2).

1.4. **Solving Thue Equations: Siegel's Identity and S-Unit Equations.** The method for solving Thue equations in both the number field and function field settings begins similarly. We specialize to the case where A denotes either the ring $\mathbb{Z}$ or the ring $\mathbb{C}[T]$. Let $(x,y) \in A^2$ denote a solution to the Thue equation

$$F(X,Y) = m, \tag{4}$$

where $F(X,Y) \in A[X,Y]$ is a homogeneous form of degree n, and $m \in A$ is non-zero. For simplicity, we moreover assume that $f(X) := F(X,1)$ is monic, so that we may factor

$$F(x,y) = (x - \alpha_1 y) \dots (x - \alpha_n y) = m, \tag{5}$$

where $\alpha_1, \dots, \alpha_n$ denote the roots of $f(X)$.

Let k denote the fraction field of A (i.e. either $\mathbb{Q}$ or $\mathbb{C}(T)$), and let K denote the splitting field of $f(X)$ over k. We moreover use $\mathcal{O}_K$ to denote the ring of integers of K, that is $\mathcal{O}_K$ denotes the integral closure of A in K. From (5) it follows that $\beta_i := x - \alpha_i y$ are *S-units* in $\mathcal{O}_K$, where S denotes the set of prime ideals in $\mathcal{O}_K$ that lie above either a prime dividing m or the prime at infinity. By *Siegel's identity* we moreover find that

$$-\frac{(\alpha_2 - \alpha_3)}{(\alpha_1 - \alpha_2)} \frac{\beta_1}{\beta_3} - \frac{(\alpha_3 - \alpha_1)}{(\alpha_1 - \alpha_2)} \frac{\beta_2}{\beta_3} = 1.$$

Upon setting $u_1 := -\frac{(\alpha_2-\alpha_3)}{(\alpha_1-\alpha_2)} \frac{\beta_1}{\beta_3}$ and $u_2 := -\frac{(\alpha_3-\alpha_1)}{(\alpha_1-\alpha_2)} \frac{\beta_2}{\beta_3}$, we thus obtain a solution to the *S-unit equation*

$$u_1 + u_2 = 1, \tag{6}$$

where $u_1, u_2 \in K$ are again S-units, where S now moreover includes the finite set of primes in K dividing $(\alpha_2 - \alpha_3), (\alpha_1 - \alpha_2)$, or $(\alpha_3 - \alpha_1)$.

In the classical setting, one may use Baker's method of lower bounds for linear forms in logarithms to obtain an effective upper bound on the height of the possible solutions to such S-unit equations. Since each solution $(x,y) \in \mathbb{Z}^2$ of the Thue equation $F(X,Y) = m$ corresponds to a pair of S-units $(u_1,u_2) \in K^2$ satisfying (6), one may effectively determine the entire set of solutions to (4).

1.5. **A $\mathbb{C}(T)$ Strategy for Solving Thue Equations: The ABC Conjecture.** One may alternatively obtain an upper bound on the height of the possible solutions to (6) by applying an appropriate form of the *ABC conjecture*. First formulated by Joseph Oesterlé and David Masser in 1985, the ABC conjecture is considered perhaps the most important unsolved problem in Diophantine analysis. The classical version may be stated as follows: let $a, b, c \in \mathbb{Z}$, such that $a + b = c$, and suppose moreover that a, b, and c are pairwise co-prime. Then for any $\epsilon > 0$, there exists a constant M_ϵ such that

$$\max(|a|, |b|, |c|) \leq M_\epsilon \prod_{p|abc} p^{1+\epsilon}.$$

Recall that the *height* of any $r \in \mathbb{Q}^\times$ is defined to be $H_\mathbb{Q}(r) := \max(\log |m|, \log |n|)$, where $r = m/n$ and $(m, n) = 1$. The ABC conjecture may thus be reformulated as follows:

Conjecture 1 (ABC)**.** Fix $\epsilon > 0$ and suppose $u + v = 1$, where $u, v \in \mathbb{Q}$. Then there exists a constant m_ϵ such that

$$\max(H_\mathbb{Q}(u), H_\mathbb{Q}(v)) \leq m_\epsilon + (1 + \epsilon) \prod_{p|abc} \log p,$$

where $u = a/c$ and $v = b/c$, and where $(a, b, c) = 1$.

An effective version of Conjecture 1 would provide an immediate means by which to solve equations of the form $u_1 + u_2 = 1$, where $u_1, u_2 \in \mathbb{Q}$ are S-units, for any finite fixed set of primes, S. More generally, an effective version of the *ABC* conjecture formulated over K, where K denotes either a number field or a function field, would enable an effective means by which to compute all solutions to (6), and thereby solve the Thue equation (4).

While such a result is currently far out of reach in the classical setting, over function fields the corresponding *ABC Theorem* is true, unconditionally. In this setting, the appropriate constant m_ϵ may moreover be explicitly computed in terms of g_K, the *genus* of K. The ABC theorem may thus be used to obtain an effective upper bound for the height of any pair of S-units $(u_1, u_2) \in K^2$ satisfying (6). As noted in [12, p. 18], the bounds this method produces in the function field setting are comparatively much smaller to those obtained in the classical setting via Baker's method.

1.6. **Structure of Paper.** The remainder of this paper is structured as follows. Section 2 provides general background on valuation theory, the ABC Theorem, and discriminants, within the $\mathbb{C}(T)$ setting. Section 3 establishes certain properties of the forms $F_\lambda(X, Y)$ in (3), as well as the roots α of the polynomial $f_\lambda(X) := F_\lambda(X, 1)$. Since a solution $(x, y) \in S_{\lambda,\xi}$ corresponds to a unit $x - \alpha y$ in the ring $\mathbb{C}[T][\alpha]$, in Section 4 we then identify a system of fundamental units for the $\mathbb{C}[T][\alpha]$. In Section 5 we then estimate the genus of K, the splitting field of $f_\lambda(X)$ over $\mathbb{C}(T)$, and apply the ABC Theorem to obtain a bound on the height of solutions to the corresponding S-unit equations. Finally in 6 we apply these bounds to prove Theorem 1, where the relevant computational details are then provided in the Appendix.

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2. Background: Valuations, the ABC Theorem, and Discriminants

2.1. **Valuations on $\mathbb{C}(T)$.** Let F denote a field. Recall that $v : F \to \mathbb{R} \cup \{\infty\}$ is said to be a *valuation* on F if the following properties hold (see e.g. [2, p. 19]):

i) $v(a) = \infty$ if and only if $a = 0$
ii) $v(ab) = v(a) + v(b)$
iii) $v(a + b) \geq \min\{v(a), v(b)\}$, and
$v(a + b) = \min\{v(a), v(b)\}$ whenever $v(a) \neq v(b)$.

We say that two valuations v_1 and v_2 are *equivalent* if there exists a constant $c > 0$ such that $v_1(f) = c \cdot v_2(f)$ for all $f \in F$. A *place* on F is then an equivalence class of (non-trivial) valuations on F. We denote the set of places on a field F by M_F. By abuse of notation we allow v to refer to both a valuation and to its corresponding place.

For $a \in \mathbb{C}$, consider the (discrete) valuation $v_a : \mathbb{C}(T) \to \mathbb{Z} \cup \{\infty\}$ obtained by setting $v_a(T - a) = 1$. We moreover consider the *valuation at infinity*, denoted v_∞, obtained by setting $v_\infty(f) = -\deg(f)$ for any $f \in \mathbb{C}[T]$. By an analogue of Ostrowski's theorem, we find that $M_{\mathbb{C}(T)} = \{v_a : a \in \mathbb{C} \cup \{\infty\}\}$.

A valuation v naturally determines a *norm* via $|a|_v := e^{-v(a)}$. This in turn induces a metric on F, whose completion we denote by F_v. Thus, we may naturally extended v to a function $v : F_v \to \mathbb{R} \cup \{\infty\}$. Note that the completion of $\mathbb{C}(T)$ with respect to v_∞ is the field of formal Laurent series in the variable $1/T$, namely

$$\mathbb{C}((1/T)) := \left\{ \sum_{n \geq n_0} a_n T^{-n} : n_0 \in \mathbb{Z}, a_i \in \mathbb{C}, a_{n_0} \neq 0 \right\} \cup \{0\}.$$

For any $z = \sum_{n \geq n_0} a_n T^{-n} \in \mathbb{C}((1/T))$ as above, we then find that $v_\infty(z) = n_0$.

Let $K/\mathbb{C}(T)$ denote a finite algebraic extension of degree n, and let $\mathcal{O}_K \subseteq K$ denote the integral closure of $\mathbb{C}[T]$ in K. To any prime ideal $\mathfrak{p}_i \subseteq \mathcal{O}_K$ one may associate a valuation on K as follows. For any $f \in K$, we consider the principal (fractional) ideal

$$(f) = \prod_{\mathfrak{p}} \mathfrak{p}^{w_\mathfrak{p}(f)}.$$

Then the map $w_\mathfrak{p} : f \mapsto w_\mathfrak{p}(f)$ defines a valuation on K.

For $a \in \mathbb{C}$, let $(T - a)\mathcal{O}_K$ denote the principal ideal in $\mathcal{O}_K$ generated by $(T - a)$, and write $(T - a)\mathcal{O}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g}$, where $\mathfrak{p}_1, \ldots, \mathfrak{p}_g \subseteq \mathcal{O}_K$ denote prime ideals. The scaled valuation $w'_{\mathfrak{p}_i} = \frac{1}{e_i} w_{\mathfrak{p}_i}$ extends v_a to a valuation on K, and we say that the place $w_{\mathfrak{p}_i}$ *lies above* the place v_a. Any place $w \in M_K$ lying above v_a, where $a \in \mathbb{C}$, is referred to as a *finite place* on K.

When $a = \infty$, we instead consider the ring $\mathbb{C}[1/T]$, and let $\mathcal{O}'_K$ denote its integral closure in K. As above, we may factor $\frac{1}{T}\mathcal{O}'_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g}$ into prime ideals in $\mathcal{O}'_K$. Each such prime ideal $\mathfrak{p}_i$ corresponds to a place $w_i \in M_K$ which extends v_∞ to a valuation on K (up to scaling). We say that the places $w_1, \ldots, w_g$ *lie above* v_∞ and refer to these as the *infinite places* on K. Every place $w \in M_K$ is found to lie above v_a for some $a \in \mathbb{C} \cup \{\infty\}$.

Each $e_i \in \mathbb{N}$ above is referred to as the *ramification index* of the corresponding prime $\mathfrak{p}_i$. The prime $(T - a)\mathbb{C}[T]$ (resp. the prime $\frac{1}{T}\mathbb{C}[1/T]$) is said to *ramify* in K whenever $e_i > 1$ for some i. We moreover find that $e_1 + \cdots + e_g = n$, and in the particular case that $K/\mathbb{C}(T)$ is Galois, we have that $e := e_1 = \cdots = e_g$, i.e. that $eg = n$.

The *product formula* states that

$$\sum_{w \in M_K} w(f) = 0 \quad \text{for any } f \in K.$$

In particular, if $\mu \in \mathcal{O}_K^\times$ is a unit, then $w(\mu) = 0$ at any finite place $w \in M_K$, from which it follows that

$$\sum_{w | v_\infty} w(\mu) = 0 \quad \text{for any } \mu \in \mathcal{O}_K^\times. \tag{7}$$

We moreover find that $w(\mu) = 0$ at all $w \in M_K$ if and only if $\mu \in \mathbb{C}^\times$.

2.2. **The $\mathbb{C}(T)$ ABC Theorem.** Let K denote a finite algebraic extension of $\mathbb{C}(T)$. Recall that the *height* of an element $f \in K^\times$ is defined to be

$$H_K(f) := -\sum_{w \in M_K} \min(0, w(f)).$$

The following theorem, a slight variation of [12, Ch. 1 Lemma 2], provides an explicit upper bound for the height of solutions to an S-unit equation. It may be viewed as a special case of the ABC-theorem for function fields:

Theorem A (ABC)**.** *Let $\gamma_1, \gamma_2 \in K$ with $\gamma_1 + \gamma_2 = 1$. Let $\mathcal{W}$ be a finite set of valuations such that for all $w \notin \mathcal{W}$ we have $w(\gamma_1) = w(\gamma_2) = 0$. Then*

$$H_K(\gamma_1) \le \max(0, 2g_K - 2 + |\mathcal{W}|),$$

where g_K is the genus of K.

The ABC Theorem is stated in terms of the genus, g_K. A bound on g_K may be obtained using the *Riemann–Hurwitz Formula* (see e.g. [17, Theorem 7.16]), which we state in the following special case:

Theorem B (Riemann–Hurwitz)**.** *Let K denote a finite algebraic extension of $\mathbb{C}(T)$. Then*

$$2g_K - 2 = [K : \mathbb{C}(T)] \cdot (-2) + \sum_{w \in M_K} (e_w - 1),$$

where e_w denotes the ramification index of $w \in M_K$.

2.3. **Discriminants.** Consider a principal ideal domain A with field of fractions F. We now recall several different notions of the *discriminant*.

Definition 1A. Let $f(X) \in F[X]$ be a monic polynomial of degree n, and suppose $f(X) = (X - \alpha_1) \cdots (X - \alpha_n)$, where $\alpha_1, \dots, \alpha_n \in \overline{F}$, the algebraic closure of F. We define the *discriminant* of f to be

$$\operatorname{disc}(f) := \prod_{i<j} (\alpha_i - \alpha_j)^2.$$

For A and F as above, let K/F denote a finite Galois extension of degree n. Let $\sigma_1, \dots, \sigma_n$ moreover denote the distinct elements of the Galois group, where we note that $|\mathrm{Gal}(K/F)| = n$, since K/F is Galois.

Definition 1B. For any $e_1, \dots, e_n \in K$ we define the *discriminant* of $(e_1, \dots, e_n)$ to be

$$\operatorname{disc}(e_1, \dots, e_n) := (\det(\sigma_i(e_j))_{i,j})^2.$$

Since K/F is finite and Galois, it is, in particular, finite and separable, and thus by the primitive element theorem we may write $K = F(\alpha)$, for some $\alpha \in K$. Let $f \in F[X]$ denote the minimal polynomial of α, and write $f(X) = (X - \alpha_1) \cdots (X - \alpha_n)$. Since K/F is Galois, every irreducible polynomial $f \in F[X]$ with a root in K splits over K and is separable. It follows that $\alpha_1, \dots, \alpha_n$ all lie in K and are distinct.

For each $\sigma \in \mathrm{Gal}(K/F)$, we find that $f(\sigma(\alpha)) = \sigma(f(\alpha)) = 0$, and therefore $\sigma(\alpha)$ is also a root of $f(X)$. Note that every σ is determined uniquely by the value of $\sigma(\alpha)$, and thus $\sigma_i(\alpha) \neq \sigma_j(\alpha)$ for $i \neq j$. Since $|\mathrm{Gal}(K/F)| = [K : F] = \deg(f) = n$, we may in fact write $\sigma_i(\alpha) := \alpha_i$ for each $1 \leq i \leq n$. We thus obtain the following relation:

$$\begin{aligned}\mathrm{disc}(1, \alpha, \dots, \alpha^{n-1}) = (\det(\sigma_i(\alpha^{j-1}))_{i,j})^2 &= \prod_{i<j} (\sigma_i(\alpha) - \sigma_j(\alpha))^2 \\ &= \prod_{i<j} (\alpha_i - \alpha_j)^2 = \mathrm{disc}(f).\end{aligned} \tag{8}$$

Here we use the fact that $(\sigma_i(\alpha^{j-1}))_{i,j} = (\sigma_i(\alpha)^{j-1})_{i,j}$ is a Vandermonde matrix, and thus its determinant is equal to $\prod_{i<j}(\sigma_i(\alpha) - \sigma_j(\alpha))$.

Let B denote the integral closure of A in K, and let $e_1, \dots, e_n \in B$ denote a basis for K/F.

Definition 1C. Consider the free A-module

$$M = \left\{ \sum_{i=1}^{n} a_i e_i : a_i \in A \right\} \subseteq B.$$

We define the *discriminant* of M, denoted $D_A(M)$, to be the principal ideal in A that is generated by $\mathrm{disc}(e_1, \dots, e_n)$. The discriminant of the field extension K/F is defined to be

$$D_{K/F} := D_A(B).$$

Note that, indeed, $\mathrm{disc}(e_1, \dots, e_n) \in A$, and moreover that $D_A(M)$ is well-defined, i.e. does not depend on our particular choice $\{e_1, \dots, e_n\}$ for a basis of M.

Lemma A. *Suppose M' be an A-submodule of M of the above form. Then $D_A(M) | D_A(M')$, i.e. $D_A(M') \subseteq D_A(M)$.*

Proof. Note that $D_A(M')$ is generated by some $\mathrm{disc}(e'_1, \dots, e'_n)$, where $e'_1, \dots, e'_n \in M' \subseteq M$. In particular, we may write $(e'_1, \dots, e'_n) = (e_1, \dots, e_n) \cdot P$ for some $P \in A^{n \times n}$. Thus $\mathrm{disc}(e'_1, \dots, e'_n) = (\det P)^2 \,\mathrm{disc}(e_1, \dots, e_n) \in D_A(M)$, and therefore $D_A(M') \subseteq D_A(M)$, as desired. □

In subsequent computations we will make use of the following important fact about discriminants. For a proof (in a more general setting) see e.g. [15, Chapter III, Corollary 2.12].

Lemma B. *A prime $\mathfrak{p} \subset A$ is ramified in B if and only if $\mathfrak{p}$ divides $D_{K/F}$.*

3. A simple quartic family over $\mathbb{C}(T)$

Consider the family of quartic, binary forms

$$F_\lambda(X, Y) := X^4 - \lambda X^3 Y - 6X^2Y^2 + \lambda XY^3 + Y^4,$$

where $\lambda \in \mathbb{C}[T]/\{\mathbb{C}\}$, and let $\mathfrak{a} := \deg \lambda > 0$. Define

$$f_\lambda(X) := F_\lambda(X, 1) = X^4 - \lambda X^3 - 6X^2 + \lambda X + 1,$$

and note that

$$F_\lambda(X, Y) = Y^4 f_\lambda\left(\frac{X}{Y}\right).$$

For $z \in \overline{\mathbb{C}(T)} \setminus \{0, \pm 1\}$, consider the rational maps

$$\phi(z) := \frac{z-1}{z+1} \quad \phi^2(z) = -\frac{1}{z} \quad \phi^3(z) = \frac{1+z}{1-z} \quad \phi^4(z) = z, \tag{9}$$

and note that $z, \phi(z), \phi^2(z), \phi^3(z)$ are distinct whenever $z \neq \pm i$. Furthermore, if α is a root of f_λ, one may check that $f_\lambda(\phi(\alpha)) = 0$, i.e. $\phi(\alpha)$ is also a root of f_λ. The four distinct roots of f_λ are thus given by $\alpha_i := \phi^{i-1}(\alpha)$ for each $1 \leq i \leq 4$ (upon noting that $\alpha \neq \pm i$).

Lemma 1. *Suppose* $\deg \lambda > 0$. *Then* $f_\lambda(X)$ *is irreducible over* $\mathbb{C}[T][X]$.

Proof. Suppose $f_\lambda(X) \in \mathbb{C}[T][X]$ is reducible. Then either $f_\lambda(X)$ contains a root $\alpha(T) \in \mathbb{C}[T]$, or $f_\lambda(X)$ factors into two quadratic polynomials. In the first case, we write $f_\lambda(X) = (X - \alpha(T))(X^3 + a(T)X^2 + b(T)X + c(T))$, where $a(T), b(T), c(T) \in \mathbb{C}[T]$. In particular, we have $\alpha(T)c(T) = 1$, which implies $\alpha := \alpha(T) \in \mathbb{C}[T]^\times = \mathbb{C}^\times$. It moreover follows from (9) that $\phi(\alpha), \phi^2(\alpha), \phi^3(\alpha) \in \mathbb{C}$. Thus all coefficients f_λ lie in $\mathbb{C}$. In particular, $\lambda \in \mathbb{C}$, contradicting our initial assumption that $\deg \lambda > 0$.

In the second case, we write $f_\lambda(X) = (X^2 + a(T)X + b(T))(X^2 + c(T)X + d(T))$, where $a(T), b(T), c(T), d(T) \in \mathbb{C}[T]$. In particular, we find that $b(T)d(T) = 1$, which implies that $b(T), d(T) \in \mathbb{C}[T]^\times = \mathbb{C}^\times$. In other words, $f_\lambda(X) = (X^2 + a(T)X + b)(X^2 + c(T)X + d)$, where $b, d \in \mathbb{C}^\times$. Equating coefficients of X^2, we then find that $-6 = a(T)c(T) + b + d$, which again implies $a(T), c(T) \in \mathbb{C}$. Since all coefficients f_λ lie in $\mathbb{C}$, it follows, in particular, that $\lambda \in \mathbb{C}$, contradicting our initial assumption.

□

Since $\alpha_i = \phi^{i-1}(\alpha) \in \mathbb{C}(T)(\alpha)$ for all $1 \leq i \leq 4$, we find that $K := \mathbb{C}(T)(\alpha)$ is the splitting field of f_λ over $\mathbb{C}(T)$. In other words, K is a normal extension, which implies K is Galois. For $\sigma \in \mathrm{Gal}(K/\mathbb{C}(T))$, we moreover note that $f_\lambda(\sigma(\alpha)) = \sigma(f_\lambda(\alpha)) = 0$, and therefore $\sigma(\alpha) = \phi^i(\alpha)$ for some $1 \leq i \leq 4$. By Lemma 1, $|\mathrm{Gal}(K/\mathbb{C}(T))| = \deg(f_\lambda) = 4$. Since σ is uniquely determined by the value of $\sigma(\alpha) \in K$, we can define each $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \mathrm{Gal}(K/\mathbb{C}(T))$ by setting $\sigma_i(\alpha) = \alpha_i$.

Let $(x, y) \in \mathbb{C}[T] \times \mathbb{C}[T]$ denote some solutions to $F_\lambda(X, Y) = \xi$, where $\xi \in \mathbb{C}^\times$. Define

$$\beta_i := x - \alpha_i y$$

and write $\beta := \beta_1 = x - \alpha y$. Since

$$\begin{aligned} F_\lambda(x, y) = y^4 f_\lambda\left(\frac{x}{y}\right) &= y^4(x/y - \alpha_1)(x/y - \alpha_2)(x/y - \alpha_3)(x/y - \alpha_4) \\ &= (x - \alpha_1 y)(x - \alpha_2 y)(x - \alpha_3 y)(x - \alpha_4 y) = \xi, \end{aligned}$$

the elements $\beta_i = x - y\alpha_i$ are units in the ring $\mathbb{C}[T][\alpha_1, \alpha_2, \alpha_3, \alpha_4]$. Conversely, any unit $\beta \in \mathbb{C}[T][\alpha_1, \alpha_2, \alpha_3, \alpha_4]$ of the form $\beta = x - \alpha y$ yields a solution $(x, y) \in S_{\lambda,\xi}$. Thus, finding the solution set $S_{\lambda,\xi}$ is equivalent to finding the set of units $\beta \in \mathbb{C}[T][\alpha_1, \alpha_2, \alpha_3, \alpha_4]^\times$ of the shape $\beta = x - \alpha y$, where $x, y \in \mathbb{C}[T]$. In fact we have the following lemma.

Lemma 2. *Let* $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ *denote the roots of* $f_\lambda(X)$. *Then* $\mathbb{C}[T][\alpha_1, \alpha_2, \alpha_3, \alpha_4] = \mathbb{C}[T][\alpha_1]$.

Proof. It suffices to demonstrate that $\alpha_2, \alpha_3, \alpha_4 \in \mathbb{C}[T][\alpha] = \{A\alpha^3 + B\alpha^2 + C\alpha + D : A, B, C, D \in \mathbb{C}[T]\}$, where $\alpha := \alpha_1$. To show that $\alpha_2 \in \mathbb{C}[T][\alpha]$, we note that $\alpha_2 = \phi(\alpha) = (\alpha - 1)/(\alpha + 1)$. Since clearly $\alpha - 1 \in \mathbb{C}[T][\alpha]$, it suffices to demonstrate that $(\alpha + 1)^{-1} \in \mathbb{C}[T][\alpha]$. Let us write

$$(\alpha + 1)^{-1} = A\alpha^3 + B\alpha^2 + C\alpha + D, \qquad A, B, C, D \in \mathbb{C}(T),$$

and note that $(\alpha+1)^{-1} \in \mathbb{C}[T][\alpha]$ if and only if $A, B, C, D \in \mathbb{C}[T]$. We then compute

$$\begin{aligned}
1 &= (\alpha+1)(A\alpha^3 + B\alpha^2 + C\alpha + D)\\
&= A\alpha^4 + (A+B)\alpha^3 + (B+C)\alpha^2 + (C+D)\alpha + D\\
&= A(\lambda\alpha^3 + 6\alpha^2 - \lambda\alpha - 1) + (A+B)\alpha^3 + (B+C)\alpha^2 + (C+D)\alpha + D\\
&= (A\lambda + A + B)\alpha^3 + (6A + B + C)\alpha^2 + (-\lambda A + C + D)\alpha + (-A + D).
\end{aligned}$$

Comparing coefficients and solving the system of equations

$$A(\lambda+1) + B = 0, \quad 6A + B + C = 0, \quad -\lambda A + C + D = 0, \quad -A + D = 1,$$

we get that

$$A = \frac{1}{4}, \quad B = \frac{-\lambda - 1}{4}, \quad C = \frac{\lambda - 5}{4}, \quad D = \frac{5}{4}.$$

It follows that

$$\frac{1}{(\alpha+1)} = \frac{1}{4}\left(\alpha^3 - (\lambda+1)\alpha^2 + (\lambda - 5)\alpha + 5\right). \tag{10}$$

Thus, $\alpha_2 = (\alpha-1)/(\alpha+1) \in \mathbb{C}[T][\alpha]$, and therefore $\mathbb{C}[T][\alpha_2] \subseteq \mathbb{C}[T][\alpha]$. By the exact same argument, we find that $\mathbb{C}[T][\alpha_3] \subseteq \mathbb{C}[T][\alpha_2]$, and also that $\mathbb{C}[T][\alpha_4] \subseteq \mathbb{C}[T][\alpha_3]$, i.e. that $\mathbb{C}[T][\alpha_2, \alpha_3, \alpha_4] \subseteq \mathbb{C}[T][\alpha]$, from which the claim then follows. □

3.1. **Computing Laurent Series of α.** The following is a corollary of Hensel's Lemma:

Lemma C. *If $f(t, X)$ is a polynomial in two variables over a field k, and $X = a$ is a simple root of $f(0, X)$, then there is a unique power series $X(t)$ with $X(0) = a$ and $f(t, X(t)) = 0$ identically.*

Proof. See [3, Corollary 7.4]. □

Lemma 3. *The polynomial $f_\lambda(X) = X^4 - \lambda X^3 - 6X^2 + \lambda X + 1$ has four distinct roots in $\mathbb{C}((1/\lambda))$, which take the following shape:*

$$\alpha = 1 - \frac{2}{\lambda} + \frac{2}{\lambda^2} + \frac{8}{\lambda^3} + \dots \qquad \alpha_2 = -\frac{1}{\lambda} + \frac{5}{\lambda^3} + \dots$$
$$\alpha_3 = -1 - \frac{2}{\lambda} - \frac{2}{\lambda^2} + \frac{8}{\lambda^3} + \dots \qquad \alpha_4 = \lambda + \frac{5}{\lambda} + \dots.$$

Proof. Note that $f_\lambda(\alpha) = 0$ if and only if $\tilde{f}(1/\lambda, \alpha) = 0$, where

$$\tilde{f}\left(\frac{1}{\lambda}, X\right) := \frac{1}{\lambda} f_\lambda(X) = \frac{1}{\lambda}X^4 - X^3 - \frac{6}{\lambda}X^2 + X + \frac{1}{\lambda} = 0.$$

Note further that $-1, 0, 1$ are each simple roots of $\tilde{f}(0, X) = -X^3 + X$. In particular, 1 is a simple root of $\tilde{f}(0, X)$. By Lemma C, there then exists a unique power series of the form $X(1/\lambda) = 1 + a_1/\lambda + a_2/\lambda^2 + \dots$, such that

$$\tilde{f}\left(\frac{1}{\lambda}, X\left(\frac{1}{\lambda}\right)\right) = 0.$$

Equivalently, $X(1/\lambda)$ is a root of $f_\lambda(X)$. Let us call this root α, i.e.

$$\alpha = 1 + \frac{a_1}{\lambda} + \frac{a_2}{\lambda^2} + \dots.$$

In order to explicitly compute the coefficients of this expansion, we note that

$$\frac{1}{\lambda}\left(1+a_1\frac{1}{\lambda}+\dots\right)^4-\left(1+a_1\frac{1}{\lambda}+\dots\right)^3-\frac{6}{\lambda}\left(1+a_1\frac{1}{\lambda}+\dots\right)^2+\left(1+a_1\frac{1}{\lambda}+\dots\right)+\frac{1}{\lambda}=0,$$

and compare coefficients. The coefficient of $1/\lambda$ on the left-hand side is equal to $1-3a_1-6+a_1+1$, which upon setting equal to 0, implies $a_1=-2$. Considering higher powers of $1/\lambda$, we similarly find that

$$\alpha = 1-\frac{2}{\lambda}+\frac{2}{\lambda^2}+\frac{8}{\lambda^3}+\dots.$$

To obtain the Laurent series representations for the other roots of $f_\lambda(X)$, we recall that $1/(1-x)=1+x+x^2+\dots$, and then compute

$$\begin{aligned}
\alpha_2=\phi(\alpha)=\frac{\alpha-1}{\alpha+1}&=\frac{-\frac{2}{\lambda}+\frac{2}{\lambda^2}+\frac{8}{\lambda^3}+\dots}{2-\frac{2}{\lambda}+\frac{2}{\lambda^2}+\frac{8}{\lambda^3}+\dots}=\frac{-\frac{1}{\lambda}+\frac{1}{\lambda^2}+\frac{4}{\lambda^3}+\dots}{1-\frac{1}{\lambda}+\frac{1}{\lambda^2}+\frac{4}{\lambda^3}+\dots}\\
&=\left(-\frac{1}{\lambda}+\frac{1}{\lambda^2}+\frac{4}{\lambda^3}+\dots\right)\frac{1}{1-(\frac{1}{\lambda}-\frac{1}{\lambda^2}-\frac{4}{\lambda^3}+\dots)}\\
&=\left(-\frac{1}{\lambda}+\frac{1}{\lambda^2}+\frac{4}{\lambda^3}+\dots\right)\left(1+(\frac{1}{\lambda}-\frac{1}{\lambda^2}-\frac{4}{\lambda^3}+\dots)+(\frac{1}{\lambda}-\frac{1}{\lambda^2}-\frac{4}{\lambda^3}+\dots)^2+\dots\right)\\
&=\left(-\frac{1}{\lambda}+\frac{1}{\lambda^2}+\frac{4}{\lambda^3}+\dots\right)\left(1+\frac{1}{\lambda}-\frac{5}{\lambda^3}+\dots\right)=-\frac{1}{\lambda}+\frac{5}{\lambda^3}+\dots.
\end{aligned}$$

The roots $\alpha_3=1/\alpha$ and $\alpha_4=-1/\alpha_2$ may then be computed similarly.

□

Above we explicitly computed the four distinct roots of f_λ in $\mathbb{C}((1/\lambda))$. Note that $\mathbb{C}((1/\lambda))$ embeds into $\mathbb{C}((1/T))$, since $\lambda=\lambda_{\mathfrak{a}}T^{\mathfrak{a}}+\dots+\lambda_0$ lies in $\mathbb{C}((1/T))$ and $|1/\lambda|_{v_\infty}<1$. Thus f_λ has four distinct roots in $\mathbb{C}((1/T))$, each of which corresponds to a unique embedding $\iota: K\hookrightarrow\mathbb{C}((1/T))$ defined by $\iota_i:\alpha\to\alpha_i$ for some $1\le i\le 4$. Each embedding then induces a valuation $w_i:K\to\mathbb{Z}\cup\{\infty\}$ given by $w_i(z)=v_\infty(\iota_i(z))$ for all $z\in K$. In particular, each w_i extends the valuation v_∞ on $\mathbb{C}(T)$, and we will see from the computations below that w_1,w_2,w_3, and w_4 are distinct, i.e. that v_∞ does not ramify over K.

For $z\in K$, we moreover define

$$(z)_\infty := (w_1(z),w_2(z),w_3(z),w_4(z)).$$

For any $z\in K$, let $z_i:=\sigma_i(z)$ for $1\le i\le 4$ denote the conjugates of z. Considering $i+j-1 \bmod 4$, we note that

$$\iota_j(\sigma_i(\alpha))=\iota_j(\phi^{i-1}(\alpha))=\phi^{i-1}(\iota_j(\alpha))=\phi^{i-1}(\alpha_j)=\alpha_{i+j-1}=\iota_{i+j-1}(\alpha),$$

and therefore that in fact $\iota_j(\sigma_i(z))=\iota_{i+j-1}(z)$ for all $z\in K$. We thus find that

$$w_j(z_i)=v_\infty(\iota_j(z_i))=v_\infty(\iota_j(\sigma_i(z)))=v_\infty(\iota_{i+j-1}(z))=w_{i+j-1}(z),$$

and conclude that, for any $i,j\in\{1,2,3,4\}$, the following sets are equal:

$$\begin{aligned}
\{w_1(z),w_2(z),w_3(z),w_4(z)\}&=\{w_1(z_i),w_2(z_i),w_3(z_i),w_4(z_i)\}\\
&=\{w_j(z_1),w_j(z_2),w_j(z_3),w_j(z_4)\}.
\end{aligned}\tag{11}$$

4. Unit Structure of $\mathbb{C}[T][\alpha]^\times$

Next, we wish to find a *system of fundamental units* for $\mathbb{C}[T][\alpha]$. Note that since $\alpha\alpha_2\alpha_3\alpha_4 = 1$, we find, in particular, that α is a unit in $\mathbb{C}[T][\alpha]$. Similarly, from (10) we know that $\alpha+1$ is a unit in $\mathbb{C}[T][\alpha]$. Finally, as α_2 is a unit, it follows that $\alpha-1=\alpha_2(1+\alpha)$ is also a unit. We wish to show that $\alpha, \alpha+1$, and $\alpha-1$ form a fundamental system for $\mathbb{C}[T][\alpha]^\times$. To this end, we proceed by computing the valuations of $\alpha, \alpha+1$, and $\alpha-1$ at the four places lying above v_∞.

Lemma 4. *We have the following valuations:*

$$(\alpha)_\infty = (0,\mathfrak{a},0,-\mathfrak{a}), \quad (\alpha-1)_\infty = (\mathfrak{a},0,0,-\mathfrak{a}), \quad (\alpha+1)_\infty = (0,0,\mathfrak{a},-\mathfrak{a}).$$

Proof. Since $v_\infty(c/\lambda^n) = n\mathfrak{a}$ for any $c \in \mathbb{C}^\times$, it follows from Lemma 3 that

$$w_1(\alpha) = v_\infty(\alpha_1) = v_\infty\left(1 - \frac{2}{\lambda} + \frac{2}{\lambda^2} + \frac{8}{\lambda^3} + \dots\right) = v_\infty(1) = 0,$$

and similarly that

$$\begin{aligned} w_2(\alpha) &= v_\infty(\alpha_2) = v_\infty\left(-\frac{1}{\lambda} + \frac{5}{\lambda^3} + \dots\right) = \mathfrak{a} \\ w_3(\alpha) &= v_\infty(\alpha_3) = v_\infty\left(-1 - \frac{2}{\lambda} - \frac{2}{\lambda^2} + \frac{8}{\lambda^3} + \dots\right) = 0 \\ w_4(\alpha) &= v_\infty(\alpha_4) = v_\infty\left(\lambda + \frac{5}{\lambda} + \dots\right) = -\mathfrak{a}. \end{aligned}$$

from which it follows that $(\alpha)_\infty = (0,\mathfrak{a},0,-\mathfrak{a})$. Moreover,

$$\begin{aligned} \alpha_1 - 1 &= -\frac{2}{\lambda} + \frac{2}{\lambda^2} + \frac{8}{\lambda^3} + \dots, & \alpha_1 + 1 &= 2 - \frac{2}{\lambda} + \frac{2}{\lambda^2} + \frac{8}{\lambda^3} + \dots, \\ \alpha_2 - 1 &= -1 - \frac{1}{\lambda} + \frac{5}{\lambda^3} + \dots, & \alpha_2 + 1 &= 1 - \frac{1}{\lambda} + \frac{5}{\lambda^3} + \dots, \\ \alpha_3 - 1 &= -2 - \frac{2}{\lambda} - \frac{2}{\lambda^2} + \frac{8}{\lambda^3} + \dots, & \alpha_3 + 1 &= -\frac{2}{\lambda} - \frac{2}{\lambda^2} + \frac{8}{\lambda^3} + \dots, \\ \alpha_4 - 1 &= \lambda - 1 + \frac{5}{\lambda} + \dots. & \alpha_4 + 1 &= \lambda + 1 + \frac{5}{\lambda} + \dots, \end{aligned}$$

from which it follows that $(\alpha-1)_\infty = (\mathfrak{a},0,0,-\mathfrak{a})$ and $(\alpha+1)_\infty = (0,0,\mathfrak{a},-\mathfrak{a})$, as desired. □

By Lemma 4 we see that $(\alpha-1)_\infty$, $(\alpha)_\infty$ and $(\alpha+1)_\infty$, are linearly independent, and therefore that α, $\alpha-1$, and $\alpha+1$ are multiplicatively independent. In other words, for any $r,s,t \in \mathbb{Z}$, we find that

$$\alpha^r(\alpha-1)^s(\alpha+1)^t = 1 \Leftrightarrow r,s,t = 0.$$

In fact, we have the following:

Proposition 1. *The units $\alpha-1$, α and $\alpha+1$ form a fundamental system for $\mathbb{C}[T][\alpha]^\times$, namely every $\varepsilon \in \mathbb{C}[T][\alpha]^\times$ can be represented as*

$$\varepsilon = \eta(\alpha-1)^r\alpha^s(\alpha+1)^t,$$

with $\eta \in \mathbb{C}^\times$ and $r,s,t \in \mathbb{Z}$.

In order to prove Proposition 1, we first prove the following lemma.

Lemma 5. *Let* $\varepsilon \in \mathbb{C}[T][\alpha]^\times$. *Then either* $\varepsilon \in \mathbb{C}^\times$ *or* $\min\{e_1, e_2, e_3, e_4\} \leq -\mathfrak{a}$, *where* $(\varepsilon)_\infty := (e_1, e_2, e_3, e_4)$.

Proof. For $\varepsilon \in \mathbb{C}[T][\alpha]^\times$, let $\varepsilon_i := \sigma_i(\varepsilon)$ for $1 \leq i \leq 4$ denote the conjugates of ε. Since ε is a unit, by (7) we find that $e_1 + e_2 + e_3 + e_4 = 0$. If $e_1 = e_2 = e_3 = e_4 = 0$, then $\varepsilon \in \mathbb{C}^\times$ and we are done. Otherwise there exists some $e_{i_0} > 0$. By (11), we moreover note that

$$\{e_1, e_2, e_3, e_4\} = \{w_2(\varepsilon_1), w_2(\varepsilon_2), w_2(\varepsilon_3), w_2(\varepsilon_4)\},$$

and thus there exists some i such that $w_2(\varepsilon_i) > 0$. From (11) it further follows that

$$\{e_1, e_2, e_3, e_4\} = \{w_1(\varepsilon_i), w_2(\varepsilon_i), w_3(\varepsilon_i), w_4(\varepsilon_i)\}$$

and thus we may replace ε by ε_i and assume, without loss of generality, that $e_2 > 0$.

Since $\varepsilon \in \mathbb{C}[T][\alpha]^\times \subset \mathbb{C}[T][\alpha]$, we can write

$$\varepsilon_i = h_0 + h_1\alpha_i + h_2\alpha_i^2 + h_3\alpha_i^3 \quad \text{for } i = 1, 2, 3, 4,$$

with $h_0, h_1, h_2, h_3 \in \mathbb{C}[T]$. We wish to solve this system of linear equations, and we do so using Cramer's rule, namely that

$$h_0 = \frac{\det A_1}{\det A},$$

where

$$A = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \alpha_1^3 \\ 1 & \alpha_2 & \alpha_2^2 & \alpha_2^3 \\ 1 & \alpha_3 & \alpha_3^2 & \alpha_3^3 \\ 1 & \alpha_4 & \alpha_4^2 & \alpha_4^3 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} \varepsilon_1 & \alpha_1 & \alpha_1^2 & \alpha_1^3 \\ \varepsilon_2 & \alpha_2 & \alpha_2^2 & \alpha_2^3 \\ \varepsilon_3 & \alpha_3 & \alpha_3^2 & \alpha_3^3 \\ \varepsilon_4 & \alpha_4 & \alpha_4^2 & \alpha_4^3 \end{pmatrix},$$

The matrix A is a Vandermonde matrix, and therefore

$$\det A = \prod_{1 \leq i < j \leq 4} (\alpha_j - \alpha_i) = (\alpha_4 - \alpha_3)(\alpha_4 - \alpha_2)(\alpha_4 - \alpha_1)(\alpha_3 - \alpha_2)(\alpha_3 - \alpha_1)(\alpha_2 - \alpha_1).$$

Hence

$$\iota_1(\det A) = (\lambda + \dots)(\lambda + \dots)(\lambda + \dots)(-1 + \dots)(-2 + \dots)(-1 + \dots) = -2\lambda^3 + \dots,$$

from which it follows that $w_1(\det A) = -3\mathfrak{a}$. Since $\iota_k : \alpha_i \mapsto \alpha_{i+k-1}$, we see, moreover, that $\iota_k(\det A) = \pm\iota_1(\det A)$. Thus $w_k(\det A) = w_1(\det A)$ for all $1 \leq k \leq 4$, and we conclude that $(\det A)_\infty = (-3\mathfrak{a}, -3\mathfrak{a}, -3\mathfrak{a}, -3\mathfrak{a})$.

If we compute $\det A_1$, we get that

$$\begin{aligned} \det A_1 &= \varepsilon_1\alpha_2\alpha_3\alpha_4(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_4)(\alpha_4 - \alpha_2) \\ &\quad - \varepsilon_2\alpha_3\alpha_4\alpha_1(\alpha_3 - \alpha_4)(\alpha_4 - \alpha_1)(\alpha_1 - \alpha_3) \\ &\quad + \varepsilon_3\alpha_4\alpha_1\alpha_2(\alpha_4 - \alpha_1)(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_4) \\ &\quad - \varepsilon_4\alpha_1\alpha_2\alpha_3(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1) \\ &= \delta - \sigma(\delta) + \sigma^2(\delta) - \sigma^3(\delta), \end{aligned}$$

where

$$\delta = \varepsilon_1\alpha_2\alpha_3\alpha_4(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_4)(\alpha_4 - \alpha_2).$$

Since $(\varepsilon)_\infty = (e_1, e_2, e_3, e_4)$, we write $\iota_1(\varepsilon_1) = c_1T^{-e_1} + \dots$, and compute

$$\begin{aligned} \iota_1(\delta) &= (c_1T^{-e_1} + \dots)(-\frac{1}{\lambda} + \dots)(-1 + \dots)(\lambda + \dots)(1 + \dots)(-\lambda + \dots)(\lambda + \dots) \\ &= -c_1T^{-e_1}\lambda^2 + \dots, \end{aligned}$$

so $w_1(\delta) = e_1 - 2\mathfrak{a}$. Similarly, we compute $\iota_2(\delta)$, $\iota_3(\delta)$ and $\iota_4(\delta)$ to obtain $w_2(\delta)$, $w_3(\delta)$ and $w_4(\delta)$. We conclude that $(\delta)_\infty = (e_1 - 2\mathfrak{a}, e_2 - 3\mathfrak{a}, e_3 - 2\mathfrak{a}, e_4 + \mathfrak{a})$.

Now for any $i = 1, 2, 3, 4$,

$$\begin{aligned} w_i(\det A_1) = w_i(\delta - \sigma(\delta) + \sigma^2(\delta) - \sigma^3(\delta)) &\geq \min\{w_i(\delta), w_i(\sigma(\delta)), w_i(\sigma^2(\delta)), w_i(\sigma^3(\delta))\} \\ &= \min\{e_1 - 2\mathfrak{a}, e_2 - 3\mathfrak{a}, e_3 - 2\mathfrak{a}, e_4 + \mathfrak{a}\}, \end{aligned}$$

where the last step follows from (11). Dividing by $\det A$ we obtain

$$\begin{aligned} w_i(h_0) = w_i\left(\frac{\det A_1}{\det A}\right) = w_i(\det A_1) - w_i(\det A) &\geq \min\{e_1 - 2\mathfrak{a}, e_2 - 3\mathfrak{a}, e_3 - 2\mathfrak{a}, e_4 + \mathfrak{a}\} + 3\mathfrak{a} \\ &= \min\{e_1 + \mathfrak{a}, e_2, e_3 + \mathfrak{a}, e_4 + 4\mathfrak{a}\}. \end{aligned}$$

Recall that $h_0 \in \mathbb{C}[T]$, and assume for the moment that $h_0 \neq 0$. Then $w_i(h_0) = v_\infty(h_0) = -\deg h_0 \leq 0$ for $i = 1, 2, 3, 4$, so $\min\{e_1 + \mathfrak{a}, e_2, e_3 + \mathfrak{a}, e_4 + 4\mathfrak{a}\} \leq 0$. Since we assume $e_2 > 0$, it follows that $\min\{e_1 + \mathfrak{a}, e_3 + \mathfrak{a}, e_4 + 4\mathfrak{a}\} \leq 0$, which implies $\min\{e_1, e_3, e_4\} \leq -\mathfrak{a}$.

Finally, we consider the case $h_0 = 0$, i.e. we assume that

$$\varepsilon = \alpha(h_1 + h_2\alpha + h_3\alpha^2),$$

where $h_1, h_2, h_3 \in \mathbb{C}[T]$. We consider two subcases, based on whether or not the following chain of equalities holds:

$$\deg h_1 = \deg h_2 + \mathfrak{a} = \deg h_3 + 2\mathfrak{a}. \tag{12}$$

Suppose first that (12) does not hold. Then

$$w_4(\varepsilon) = w_4(\alpha) + w_4(h_1 + h_2\alpha + h_3\alpha^2) = -\mathfrak{a} + w_4(h_1 + h_2\alpha + h_3\alpha^2) \leq -\mathfrak{a},$$

and we are done. Note that for the last inequality we used the following two facts: First, for any valuation v and any elements a, b, c we have $v(a + b + c) \leq \max\{v(a), v(b), b(c)\}$ so long as $v(a), v(b), v(c)$ are not all equal. Second, $w_4(h_1) = -\deg h_1$, $w_4(h_2\alpha) = -\deg h_2 - \mathfrak{a}$, $w_4(h_3\alpha^2) = -\deg h_3 - 2\mathfrak{a}$ are each ≤ 0 and the three numbers are not all equal, since we are assuming that (12) does not hold.

Suppose next that (12) does hold. Then

$$w_1(\varepsilon) = w_1(\alpha) + w_1(h_1 + h_2\alpha + h_3\alpha^2) = 0 + w_1(h_1 + h_2\alpha + h_3\alpha^2).$$

By (12) we have $w_1(h_1) = -\deg h_1 = -\deg h_3 - 2\mathfrak{a}$, $w_1(h_2\alpha) = -\deg h_2 = -\deg h_3 - \mathfrak{a}$, $w_1(h_3\alpha^2) = -\deg h_3$, which are all distinct. Thus we obtain

$$\begin{aligned} w_1(\varepsilon) = w_1(h_1 + h_2\alpha + h_3\alpha^2) &= \min\{-\deg h_3 - 2\mathfrak{a}, -\deg h_3 - \mathfrak{a}, -\deg h_3\} \\ &= -\deg h_3 - 2\mathfrak{a} \leq -\mathfrak{a}, \end{aligned}$$

and we are done.

□

Proof of Proposition 1. Let $\varepsilon \in \mathbb{C}[T][\alpha]^\times$ be an arbitrary unit. Recall that $(\alpha - 1)_\infty = (\mathfrak{a}, 0, 0, -\mathfrak{a})$, $(\alpha)_\infty = (0, \mathfrak{a}, 0, -\mathfrak{a})$ and $(\alpha + 1)_\infty = (0, 0, \mathfrak{a}, -\mathfrak{a})$. Clearly, we can multiply ε with powers of $\alpha - 1, \alpha, \alpha + 1$ to obtain a new unit of the form $\varepsilon' = \varepsilon(\alpha - 1)^r\alpha^s(\alpha + 1)^t$, where $(\varepsilon')_\infty = (e_1', e_2', e_3', e_4')$ is such that $\mathfrak{a} \leq e_1' < 2\mathfrak{a}$ and $-\mathfrak{a} < e_2', e_3' \leq 0$. Since $e_1' + e_2' + e_3' + e_4' = 0$, we have $e_4' = -e_1' - e_2' - e_3'$ and therefore $e_4' > -\mathfrak{a}$. It follows that $\min\{e_1', e_2', e_3', e_4'\} > -\mathfrak{a}$. But then Lemma 5 implies that $\varepsilon' \in \mathbb{C}^\times$, so

$$\varepsilon = \varepsilon'(\alpha - 1)^{-r}\alpha^{-s}(\alpha + 1)^{-t}, \quad \varepsilon' \in \mathbb{C}^\times,$$

as desired. □

5. Applying the ABC Theorem

5.1. Computing $D_{K/\mathbb{C}(T)}$ and Estimating g_K.

Lemma 6. *Let r_K denote the number of places $v \in M_{\mathbb{C}(T)}$ which ramify in K. Then $r_K \leq 2\mathfrak{a}$.*

Proof. Since α is integral over $\mathbb{C}[T]$, we have that $\mathbb{C}[T][\alpha] \subseteq \mathcal{O}_K$, where $\mathcal{O}_K$ denotes the integral closure of $\mathbb{C}[T]$ in K. Upon noting that $\mathbb{C}[T][\alpha]$ is a $\mathbb{C}[T]$-module with basis $\{1, \alpha, \alpha^2, \alpha^3\}$, it follows from Lemma A that the discriminant $D_{K/\mathbb{C}(T)}$ divides the discriminant $D_{\mathbb{C}[T]}(\mathbb{C}[T][\alpha])$. By (8) we then compute

$$D_{\mathbb{C}[T]}(\mathbb{C}[T][\alpha]) = \operatorname{disc}(1, \alpha, \alpha^2, \alpha^3)\mathbb{C}[T] = \operatorname{disc}(f_\lambda)\mathbb{C}[T] = 4(\lambda^2 + 16)^3\mathbb{C}[T].$$

By Lemma B, a prime $(T - a) \subset \mathbb{C}[T]$ can only ramify in K if it divides $(\lambda^2 + 16)$, i.e. if a is a root of $\lambda^2 + 16$. Since $\deg \lambda = \mathfrak{a}$, there are at most $2\mathfrak{a}$ such primes. Since, moreover, we have already seen that v_∞ does not ramify, we conclude that there are at most $2\mathfrak{a}$ primes that ramify, as desired. □

Now we can use the Riemann–Hurwitz formula to bound the genus of K, which will then be applied in ABC's Theorem.

Lemma 7. *Let r_K denote the number of places in $\mathbb{C}(T)$ which ramify in K, and let g_K denote the genus of K. Then*

$$g_K \leq \frac{3}{2} r_K - 3 \leq 3\mathfrak{a} - 3.$$

Proof. Since $[K : \mathbb{C}(T)] = 4$ and the ramification index of each ramified prime is at most 4, it follows from the Riemann–Hurwitz Formula that

$$\begin{aligned} 2g_K - 2 &= [K : \mathbb{C}(T)] \cdot (-2) + \sum_{w \in M_K} (e_w - 1) \\ &\leq 4(-2) + r_K(4 - 1), \end{aligned}$$

which implies $g_K \leq 3r_K/2 - 3$. The second inequality now follows by Lemma 6. □

5.2. Application of the ABC Theorem.

In what follows, we use the ABC Theorem to first estimate the height $(\alpha_2 - \alpha_3)\beta_1/(\alpha_3 - \alpha_1)\beta_2$, which we in turn use to bound the height of β.

Lemma 8. *We have that*

$$H_K\left(\frac{(\alpha_2 - \alpha_3)\beta_1}{(\alpha_3 - \alpha_1)\beta_2}\right) \leq 10\mathfrak{a} - 4.$$

Proof. By Siegel's identity,

$$\begin{aligned} &\beta_1(\alpha_2 - \alpha_3) + \beta_2(\alpha_3 - \alpha_1) + \beta_3(\alpha_1 - \alpha_2) \\ &= (x - \alpha_1 y)(\alpha_2 - \alpha_3) + (x - \alpha_2 y)(\alpha_3 - \alpha_1) + (x - \alpha_3 y)(\alpha_1 - \alpha_2) = 0, \end{aligned}$$

which further implies that

$$-\frac{(\alpha_2 - \alpha_3)\beta_1}{(\alpha_3 - \alpha_1)\beta_2} - \frac{(\alpha_1 - \alpha_2)\beta_3}{(\alpha_3 - \alpha_1)\beta_2} = 1.$$

Applying Theorem A, we then obtain that

$$H_K\left(\frac{(\alpha_2 - \alpha_3)\beta_1}{(\alpha_3 - \alpha_1)\beta_2}\right) \leq \max(0, 2g_K - 2 + |\mathcal{W}|), \tag{13}$$

where $\mathcal{W}$ denotes the set of valuations $w \in M_K$ for which either

$$w\left(\frac{(\alpha_2-\alpha_3)\beta_1}{(\alpha_3-\alpha_1)\beta_2}\right) \neq 0 \quad \text{or} \quad w\left(\frac{(\alpha_1-\alpha_2)\beta_3}{(\alpha_3-\alpha_1)\beta_2}\right) \neq 0.$$

We bound the size of $|\mathcal{W}|$ from above, by counting the number of valuations for which either

$$w\left((\alpha_2-\alpha_3)\beta_1\right) \neq 0 \quad \text{or} \quad w\left((\alpha_3-\alpha_1)\beta_2\right) \neq 0 \quad \text{or} \quad w\left((\alpha_1-\alpha_2)\beta_3\right) \neq 0. \tag{14}$$

Since $(\alpha_2-\alpha_3)\beta_1, (\alpha_3-\alpha_1)\beta_2, (\alpha_1-\alpha_2)\beta_3 \in \mathcal{O}_K$, we find that

$$w\left((\alpha_2-\alpha_3)\beta_1\right), w\left((\alpha_3-\alpha_1)\beta_2\right), w\left((\alpha_1-\alpha_2)\beta_3\right) \geq 0$$

at every finite place $w \in M_K$. Hence, (14) holds at a given valuation $w \in M_K$ if and only if

$$w\left((\alpha_2-\alpha_3)\beta_1(\alpha_3-\alpha_1)\beta_2(\alpha_1-\alpha_2)\beta_3\right) > 0.$$

Since the β_i are moreover units, and $\operatorname{disc}(f_\lambda) = \prod_{1\leq i<j\leq 4}(\alpha_i-\alpha_j)^2$, we have that

$$(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)\beta_1\beta_2\beta_3 \mid \operatorname{disc}(f_\lambda) = 4(\lambda^2+16)^3.$$

Note that there are at most $2\mathfrak{a}+1$ distinct valuations $v \in M_{\mathbb{C}(T)}$ such that $v(\operatorname{disc}(f)) \neq 0$. Therefore,

$$|\mathcal{W}| \leq 2r_K + 4(2\mathfrak{a}+1-r_K) = 4+8\mathfrak{a}-2r_K.$$

Here we use the fact that if v ramifies, then there are at most 2 distinct valuations lying above v, while if v is unramified then there are exactly 4.

Finally, from (13) and the bound for g_K provided in Lemma 7, we conclude that

$$H_K\left(\frac{(\alpha_2-\alpha_3)\beta_1}{(\alpha_3-\alpha_1)\beta_2}\right) \leq 2\left(\frac{3}{2}r_K-3\right)-2+4+8\mathfrak{a}-2r_K = -4+8\mathfrak{a}+r_K \leq 10\mathfrak{a}-4,$$

as desired. □

6. Proof of Theorem 1

6.1. Bounding the Height of β. Since $(\alpha_2-\alpha_3)/(\alpha_3-\alpha_1)$ is fixed, we can next bound the height of the unit β_1/β_2.

Lemma 9. *We have that*

$$H_K\left(\frac{\beta_1}{\beta_2}\right) \leq 11\mathfrak{a}-4.$$

Proof. Let us denote the *local height* by

$$H_a(f) := -\sum_{w\mid v_a} \min(0, w(f)), \quad a \in \mathbb{C}\cup\{\infty\}.$$

Then

$$H_K(f) = \sum_{a\in\mathbb{C}\cup\{\infty\}} H_a(f) \geq H_\infty(f), \tag{15}$$

and since $w(fg) = w(f)+w(g)$ for each valuation, it follows that

$$H_a(fg) \leq H_a(f)+H_a(g)$$

for any $f, g \in K$. Moreover, since β_1/β_2 is a unit in $\mathcal{O}_K$, we have

$$H_K\left(\frac{\beta_1}{\beta_2}\right) = H_\infty\left(\frac{\beta_1}{\beta_2}\right) \leq H_\infty\left(\frac{(\alpha_2-\alpha_3)\beta_1}{(\alpha_3-\alpha_1)\beta_2}\right) + H_\infty\left(\frac{\alpha_3-\alpha_1}{\alpha_2-\alpha_3}\right). \tag{16}$$

In order to compute the last height in the above estimation, we recall that

$$\alpha_1 = 1 + \dots, \quad \alpha_2 = -\frac{1}{\lambda} + \dots, \quad \alpha_3 = -1 + \dots, \quad \alpha_4 = \lambda + \dots.$$

Therefore

$$w_1\left(\frac{\alpha_3 - \alpha_1}{\alpha_2 - \alpha_3}\right) = w_1(\alpha_3 - \alpha_1) - w_1(\alpha_2 - \alpha_3) = w_1(2 + \dots) - w_1(1 + \dots) = 0.$$

Similarly, $\iota_2(\alpha_3 - \alpha_1) = \alpha_4 - \alpha_2 = \lambda + \dots$, i.e. $w_2(\alpha_3 - \alpha_1) = -\mathfrak{a}$, and $\iota_2(\alpha_2 - \alpha_3) = \alpha_3 - \alpha_4 = -\lambda + \dots$, i.e. $w_2(\alpha_2 - \alpha_3) = -\mathfrak{a}$, which together yields

$$w_2\left(\frac{\alpha_3 - \alpha_1}{\alpha_2 - \alpha_3}\right) = -\mathfrak{a} - (-\mathfrak{a}) = 0.$$

Finally, we compute $w_3((\alpha_3 - \alpha_1)/(\alpha_2 - \alpha_3)) = 0 - (-\mathfrak{a}) = \mathfrak{a}$, and $w_4((\alpha_3 - \alpha_1)/(\alpha_2 - \alpha_3)) = -\mathfrak{a} - 0 = -\mathfrak{a}$. It follows that

$$\left(\frac{\alpha_3 - \alpha_1}{\alpha_2 - \alpha_3}\right)_\infty = (0, 0, \mathfrak{a}, -\mathfrak{a}),$$

and therefore that

$$H_\infty\left(\frac{\alpha_3 - \alpha_1}{\alpha_2 - \alpha_3}\right) = \mathfrak{a}. \tag{17}$$

By inequality (16), followed by (15) and (17), and finally Lemma 8, we conclude that

$$H_K\left(\frac{\beta_1}{\beta_2}\right) \le H_\infty\left(\frac{(\alpha_2 - \alpha_3)\beta_1}{(\alpha_3 - \alpha_1)\beta_2}\right) + H_\infty\left(\frac{\alpha_3 - \alpha_1}{\alpha_2 - \alpha_3}\right) \le H_K\left(\frac{(\alpha_2 - \alpha_3)\beta_1}{(\alpha_3 - \alpha_1)\beta_2}\right) + \mathfrak{a} \le 11\mathfrak{a} - 4,$$

as desired. □

Finally, we obtain a bound for the height of β.

Lemma 10. *We have that*

$$H_K(\beta) \le 11\mathfrak{a} - 4.$$

Proof. In the previous Lemma we obtained an upper bound for the height $H_K(\beta_1/\beta_2)$. Now we express it in a different way using the fact that $w_i(\beta_2) = w_i(\sigma(\beta_1)) = w_{i+1}(\beta_1)$ (where, as always, $i + 1$ is considered mod 4):

$$\begin{aligned} H_K\left(\frac{\beta_1}{\beta_2}\right) &= -\sum_{i=1}^{4} \min(0, w_i(\beta_1/\beta_2)) = -\sum_{i=1}^{4} \min(0, w_i(\beta_1) - w_i(\beta_2)) \\ &= \sum_{i=1}^{4} \max(0, w_i(\beta_2) - w_i(\beta_1)) = \sum_{i=1}^{4} \max(0, w_{i+1}(\beta_1) - w_i(\beta_1)). \end{aligned}$$

In order to compute this sum, let us define b_1, b_2, b_3, b_4 such that

$$\{b_1, b_2, b_3, b_4\} = \{w_1(\beta), w_2(\beta), w_3(\beta), w_4(\beta)\} \quad \text{and} \quad b_1 \le b_2 \le b_3 \le b_4.$$

Let ψ be the permutation that maps the coefficients $\{1, 2, 3, 4\}$ of the $w(\beta)$'s to the coefficients of the b's, i.e. $\psi\colon \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ such that

$$w_i(\beta) = b_{\psi(i)}, \quad i = 1, 2, 3, 4.$$

Next, we want to have a map φ for the coefficients of the b's such that if $b_i = w_j(\beta)$, then $b_{\varphi(i)} = w_{j+1}(\beta)$. Therefore, we define $\varphi\colon \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$,

$$\varphi(i) = \psi(\psi^{-1}(i) + 1).$$

Since ψ is a bijection and $j \mapsto j+1 \pmod 4$ is a 4-cycle, it is clear that φ is also a 4-cycle. Note that there exist 6 different 4-cycles.

Now we can use this notation to rewrite $H_K(\beta_1/\beta_2)$ and compute it:

$$\begin{aligned} H_K\left(\frac{\beta_1}{\beta_2}\right) &= \sum_{j=1}^{4} \max(0, b_{\varphi(j)} - b_j) \\ &= \begin{cases} b_4 - b_1 & \text{if } \varphi \in \{(1234), (1243), (1342), (1432)\}, \\ b_4 - b_1 + b_3 - b_2 & \text{if } \varphi \in \{(1324), (1423)\}. \end{cases} \end{aligned}$$

In any case,

$$H_K\left(\frac{\beta_1}{\beta_2}\right) \geq b_4 - b_1,$$

which together with Lemma 9 yields

$$b_4 - b_1 \leq 11\mathfrak{a} - 4.$$

Note that $H_K(\beta) = H_K(\beta^{-1})$ by the product formula, and thus we may assume that either $b_1 < 0$ and $0 \leq b_2 \leq b_3 \leq b_4$ or $b_1 \leq b_2 < 0$ and $0 \leq b_3 \leq b_4$ (otherwise just consider β^{-1} instead of β).

Case 1: $b_1 < 0$ and $0 \leq b_2 \leq b_3 \leq b_4$. Then we obtain

$$H_K(\beta) = -b_1 \leq -b_1 + b_4 \leq 11\mathfrak{a} - 4.$$

Case 2: $b_1 \leq b_2 < 0$ and $0 \leq b_3 \leq b_4$. Note that $2(-b_2) \leq -b_1 - b_2 = b_3 + b_4 \leq 2b_4$, so $-b_2 \leq b_4$. Thus we obtain

$$H_K(\beta) = (-b_1) + (-b_2) \leq -b_1 + b_4 \leq 11\mathfrak{a} - 4.$$

In both cases we have proven the required upper bound. □

6.2. **Completion of Proof.** Finally, we proceed to the proof of Theorem 1.

Proof of Theorem 1. Since $\beta \in \mathbb{C}[T][\alpha]^\times$ is a unit, by Proposition 1 it can be written as

$$\beta = \eta(\alpha - 1)^r \alpha^s (\alpha + 1)^t,$$

with $\eta \in \mathbb{C}^\times$ and $r, s, t \in \mathbb{Z}$. Thus, together with Lemma 10 we obtain

$$\begin{aligned} 11\mathfrak{a} - 4 \geq H_K(\beta) &= -\sum_{i=1}^{4} \min(0, w_i(\eta(\alpha - 1)^r \alpha^s (\alpha + 1)^t)) \\ &= \sum_{i=1}^{4} \max(0, -(w_i(\eta) + r w_i(\alpha - 1) + s w_i(\alpha) + t w_i(\alpha + 1))). \end{aligned}$$

Note that $w_i(\eta) = 0$ for $i = 1, 2, 3, 4$, and recall that $(\alpha - 1)_\infty = (\mathfrak{a}, 0, 0, -\mathfrak{a})$, $(\alpha)_\infty = (0, \mathfrak{a}, 0, -\mathfrak{a})$ and $(\alpha + 1)_\infty = (0, 0, \mathfrak{a}, -\mathfrak{a})$. It follows that

$$11\mathfrak{a} - 4 \geq H_K(\beta) = \max(0, -r\mathfrak{a}) + \max(0, -s\mathfrak{a}) + \max(0, -t\mathfrak{a}) + \max(0, (r + s + t)\mathfrak{a}).$$

This implies

$$\max(0, -r) + \max(0, -s) + \max(0, -t) + \max(0, r + s + t) \leq 11 - \frac{4}{\mathfrak{a}} < 11. \tag{18}$$

In particular, for each $(r, s, t) \in \mathbb{Z}^3$ which satisfies the above inequality, we have that $|r|, |s|, |t| \leq 10$. This is a (sufficiently small) finite set of values, and it remains to check which of the corresponding units $\beta = \eta(\alpha - 1)^r \alpha^s (\alpha + 1)^t \in \mathbb{C}[T][\alpha]^\times$ yield a solution $(x, y) \in S_{\lambda,\xi}$. In particular, while a general unit is of the form $\beta = x_3\alpha^3 + x_2\alpha^2 + x_1\alpha + x_0$, where $x_0, x_1, x_2, x_3 \in \mathbb{C}[T]$, we are interested in those units for which $x_3 = x_2 = 0$, i.e. units of the form $\beta = x - \alpha y$, where $x, y \in \mathbb{C}[T]$. We implement these computations using Sage [18], a code which is provided in the Appendix below. In doing so, we find that the only relevant values $(r, s, t) \in \mathbb{Z}^3$ lie in the trivial set $\{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\}$. Therefore, $\beta = x - \alpha y$ must lie in the set

$$\{\eta, \eta(\alpha-1), \eta\alpha, \eta(\alpha+1) : \eta \in \mathbb{C}^\times\} = \{\eta - \alpha \cdot 0, -\eta - \alpha(-\eta), 0 - \alpha(-\eta), \eta - \alpha(-\eta) : \eta \in \mathbb{C}^\times\}$$

which implies that

$$\begin{aligned}(x, y) \in &\{(\eta, 0), (-\eta, -\eta), (0, -\eta), (\eta, -\eta) : \eta \in \mathbb{C}^\times\} \\ &= \{(\eta, 0), (\eta, \eta), (0, \eta), (\eta, -\eta) : \eta \in \mathbb{C}^\times\}.\end{aligned}$$

We have shown that any possible solution $(x, y) \in \mathcal{S}_{\lambda,\xi}$ must lie in the above set. Plugging into $F_\lambda(X, Y) = \xi$, we find that the full solution set is indeed

$$\mathcal{S}_{\lambda,\xi} = \{(\eta, 0), (0, \eta) : \eta^4 = \xi\} \cup \{(\eta, \eta), (\eta, -\eta) : -4\eta^4 = \xi\},$$

as desired.

□

Appendix

The following Sage code outputs the units $\beta = \eta(\alpha - 1)^r \alpha^s (\alpha + 1)^t \in \mathbb{C}[T][\alpha]^\times$ such that $(r, s, t) \in \mathbb{Z}^3$ satisfy (18) and such that β is of the form $\beta = x - \alpha y$, for $x, y \in \mathbb{C}[T]$. The code may be run in less than a minute on a standard computer. Note that although the computations technically take place in an extension of $\mathbb{Q}(\ell)$ (where ℓ is a stand-in for λ) they are exactly the same as when performed in $\mathbb{C}(T)(\alpha)$.

```
F.<l> = FunctionField(QQ)
R.<x> = F[]
L.<alpha> = F.extension(x^4 -l*x^3-6*x^2+l*x+1)

for r in range(-10, 10 + 1):
    for s in range(-10 + max(0, -r), 10 - max(0, r) + 1):
        for t in range(-10 + max(0, -r) + max(0, -s),
                       10 - max(0, r) - max(0, s) + 1):
            beta = (alpha-1)^r * alpha^s * (alpha+1)^t
            betacoeff = beta.matrix()[0]
            if betacoeff[3] == 0 and betacoeff[2] == 0:
                print(r,s,t)
```

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B. Faye, UFR SATIC, Université Alioune Diop de Bambey, Diourbel, Bambey 30, Sénégal
Email address: `bernadette.faye@uadb.edu.sn`

I. Vukusic, University of Salzburg, Hellbrunnerstrasse 34/I, A-5020 Salzburg, Austria
Email address: `ingrid.vukusic@plus.ac.at`

E. Waxman, University of Haifa, Department of Mathematics, 199 Aba Khoushy Ave., Mt. Carmel, Haifa, 3498838
Email address: `ezrawaxman@gmail.com`

V. Ziegler, University of Salzburg, Hellbrunnerstrasse 34/I, A-5020 Salzburg, Austria
Email address: `volker.ziegler@plus.ac.at`